NON-LINEAR $\phi$-MODEL AND $\mathbb{C}P^{n-1}$ AT $2 + \epsilon$ DIMENSIONS

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Received 1 April 1980

Two systems, $O(n)$ non-linear $\phi$-model and $\mathbb{C}P^{n-1}$, are studied in the light of Eilenberg's theorem, on the disappearance of infrared singularities at two dimensions. The consequences of the theorem are expressed in dimensional regularization, and issues like the proper analytic continuation to $d = 2 + \epsilon$, the peculiarities of momentum-space Green functions near $d = 2$ and their renormalization, and the exponentiation of Green functions are clarified.

The analysis is applied to compute the renormalization constants, and the gauge-invariant critical exponent $\eta$ associated with the wave function of $\mathbb{C}P^{n-1}$ at one order higher than previously done. Finally, we conjecture on a possible connection between infrared finiteness and renormalizability.

1. Introduction

The purpose of this article is to indicate that what was previously considered a hindrance [1, 3, 4], the infrared divergences in the perturbation expansion of field theories with continuous symmetry near two dimensions, can be turned to good use. Once the perturbation series is cured [2], it provides a more efficient way for the computation of renormalization constants at higher orders, and thus of higher-order terms in $\epsilon = d - 2$ in the anomalous dimensions (critical exponents). Beyond this technical point it provides an insight into the structure of this type of theory.

Our initial motivation was an uneasiness about the procedure by which the exponent $\eta$ (the anomalous dimension of the field) was computed for the non-linear $\phi$-model near two dimensions [3]. The artificial nature of the calculation was already commented on in the context of the anisotropic non-linear $\phi$-model [1], where another ad hoc prescription was introduced, more useful, but no more satisfactory.

The malaise may be stated briefly as follows: the anomalous dimensions of the field $\pi$, $\frac{1}{2} \eta$, is defined via

$$\Gamma_{\pi}(p) = C p^{2-\eta}, \quad \text{for} \; T = T_c.$$
where $\Gamma_{\pi\pi}$ is the vertex associated with the Green function $\langle \pi(k)\pi(-k) \rangle$, $T_c(=O(d-2))$ is the critical temperature, and $\eta = O(\epsilon)$. One then expects that $p^{2-\eta}$ will be $p^2$ times an infinite sum of powers of $\epsilon \ln p$. This expectation is not fulfilled. In fact, up to the order of one loop the renormalized vertex reads [3]

$$\Gamma_{\pi\pi}(p) = \frac{1}{t} (p^2 + h) - \frac{1}{2} \left[ p^2 + \frac{1}{2}(n-1)h \right] \ln h, \quad (1.1)$$

where $t$ and $h$ are respectively, the renormalized temperature and external magnetic field, which also serves as an infrared regulator.

Hence, not only is the $p^2(\ln p)$ absent, but (1.1) has no limit as $h \to 0$, either at $T_c$ or below. The exponent $\eta$ is obtained by using scaling arguments, or by inserting, mechanically, the values of the fixed point into a renormalization group equation. This difficulty becomes particularly acute in a situation of crossover [1], where the value of a fixed point cannot be substituted.

The same problem may be stated in coordinate space, where $\langle \pi(p)\pi(-p) \rangle \sim p^{-(2-\eta)}$ implies $\langle \pi(x)\pi(0) \rangle \sim x^{-(\epsilon+\eta)}$; hence a power series in $\epsilon \ln x$. No such power series appears.

The advent of Elitzur's theorem [2], foreshadowed by Jevicki [5], has promised a rationalization of the above situation. The theorem asserts that an expectation value of any operator which is invariant under the symmetry group of the system ($O(n)$ in the present case) has a finite perturbation expansion, in the limit of vanishing infrared regulating mass.

While the proof of the theorem is not yet complete, there are very good reasons to believe (a) that the theorem is true [7, 2], and (b) that the strategy chosen by Elitzur is sound in its main lines. The backbone of the proof is the assertion that the perturbation expansion of the expectation values of invariant operators, is itself invariant, at every order in the expansion, under a shift of all free propagators by a constant. The corrections vanish with the regulating mass.

This fact eliminates one source of infrared problems, those associated with the appearance of singularities in the propagator in coordinate space in two dimensions. Namely,

$$G_0(x) \equiv \int d^2p \frac{e^{ip \cdot x}}{p^2 + m^2} \quad (1.2)$$

has a $\ln m$ divergence as $m \to 0$. The other side of the coin is the observation that $G_0(0) = p^{-2}$ is not a well-defined distribution in two dimensions. We will elaborate on this statement in sects. 2–4. If every $G_0(x)$ can be shifted, $G_0(0)$ can be subtracted, and the $\ln m$ eliminated.

The second line of Elitzur's attack is to show that the integrals in the graphs with the subtracted propagators do not generate new IR divergences.
We will not provide the missing elements in the proof. Instead we will employ
the theorem in two situations. The first is the non-linear $\sigma$-model, the second the
SU($N$) non-linear $\sigma$-model, or CP$^{(n-1)}$ model [6]. In both cases we will discuss the
application of the theorem using dimensional regularization in the ultraviolet, in
contrast to the lattice regularization employed by Elitzur.

The same procedure was used recently by McKane and Stone [7] to treat the
O($\sigma$) non-linear $\sigma$-model as well as the G $\otimes$ G model. We find, as they did, that
the problem by dimensional regularization facilitates the computations
enormously. It allows one to convert Elitzur's approach into an efficient practical
tool. In fact it allows the calculation of the wave function renormalization constant
at one loop higher than in previous calculations without any extra effort. If the
$\beta$-function can be computed independently to the same, higher, order, as is, for
example, the case in CP$^{(n-1)}$ [9], then all exponents can be computed to higher
order in $\epsilon$, with no extra work beyond that invested in previous computations.
Furthermore, since the computation is carried out at $h \to 0$, the proper powers of
$\epsilon \ln p$, (or $\epsilon \ln x$), reappear.

Since the computations for the non-linear $\sigma$-model was presented by McKane
and Stone [7], it is discussed very briefly here. However, there are a few important
points which were left obscure, and those are treated more extensively below. They
raise the possibility of a deeper connection between renormalizability and IR
finiteness.

The first such point is the condition $G(x = 0) = 0$, where $G$ is the $\pi\pi$
propagator in the massless limit. This condition supplements the usual rule of dimensional
regularization, by which integrals of polynomials vanish. In sect. (2) the origin of
this condition is made clear.

Once this condition is imposed, very many graphs disappear, making the compu-
tation so much easier (see sects. 4, 7). Yet once it is imposed, no IR problems are
detectable. One seems to be able to apply it to non-invariant functions. The price is
that such a function will not be renormalizable. This is shown in an example in
sect. 5, and its potential generality discussed.

The second point, left wanting in ref. [7], is the mysterious fact that the
procedure seems to work in coordinate space only. It is shown in sect. 3 that this is
not the case. Momentum space has not been lost.

Then we go to apply the technique beyond the non-linear $\sigma$-model. We define an
invariant two-point function for CP$^{(n-1)}$ in $d = 2 + \epsilon$. It is computed to O($r^3$
(O($g^2$)), and shown to be renormalizable in its dimensionally regularized form.
Then $\beta$ and $Z$ are computed to O($r^3$). To obtain the fixed point at O($r^3$) we need $\beta$
at O($r^4$); this we find in Hikami [9], who uses auxiliary information on CP$^{(1)}$, on
CP$^{(-1)}$ and on CP$^{(n-1)}$ for $n \to \infty$. Thus we obtain $\eta$, a gauge-invariant exponent,
to O($\epsilon^2$), without calculating a single integral beyond those of the $\sigma$-model at the
order of two loops. This is a new result.

Finally, we discuss some possible further applications.
2. Dimensional regularization of invariant operators

The lattice regularization used by Elitzur [2], is clearly very cumbersome and dimensional regularization looks attractive. This is even more so for systems with local gauge invariance. There is, however, a problem. In performing dimensional regularization one has to go to a dimension where integrals are convergent in the ultraviolet (i.e., \( d < 2 \)), and then analytically continue to the values of \( d \) of interest (i.e., \( d > 2 \)) [10]. This can be done either when there is an infrared cutoff, or for invariant operators, for which such cutoff is unnecessary. Yet at \( d < 2 \) even the free propagator in coordinate space,

\[
G_0(x) = \int \frac{dp}{(2\pi)^d} \left( p^2 + \mu^2 \right)^{-1} e^{-ip\cdot x},
\]

(2.1)

is undefined as \( \mu \to 0 \).

If, as in asserted by Elitzur [2], all free propagators in the expectation value of an invariant operator, in the massless limit, can be subtracted by a constant \( G_0(x = 0) \), for example) then the basic question of dimensional regularization applies to the function

\[
D(x, \mu, \epsilon) \equiv \int \frac{dp}{(2\pi)^d} \left( p^2 + \mu^2 \right)^{-1} \left[ e^{-ip\cdot x} - 1 \right],
\]

(2.2)

where \( \epsilon = d - 2 \).

The function \( D(x, \mu, \epsilon) \) is an analytic function of \( \epsilon \), for \( \epsilon < 0 \). So is \( D(x, \epsilon) = D(x, 0, \epsilon) \), which can be continued to all \( \epsilon \) in the complex plane. The continuation is performed via

\[
D(x, \mu, \epsilon) \underset{\mu \to 0}{\to} D(x, \epsilon) = \left[ (2\pi)^d S_\epsilon \epsilon^{\epsilon} \right]^{-1},
\]

(2.3)

for all \( \epsilon < 0 \), with

\[
S_\epsilon = 2\pi^{d/2} / \Gamma\left(\frac{1}{2} d\right)
\]

(2.4)

The r.h.s. of (2.3) is the analytic continuation of \( D(x, \epsilon) \) to all \( \epsilon \).

On the other hand one, can consider \( G_0 \), of (2.1) for \( \epsilon > 0 \) (\( d > 2 \)). The limit \( \mu \to 0 \) exists for all \( \epsilon > 0 \), and the oscillating exponential provides a UV cutoff. Thus,

\[
G_0(x) \to \left[ (2\pi)^d S_\epsilon \epsilon^{\epsilon} \right]^{-1},
\]

(2.5)

for \( \epsilon > 0 \), as \( \mu \to 0 \). This justifies the procedure of McKane and Stone [7]. Namely, given that all propagators are subtracted, one can replace all propagators by (2.5), and add the prescription that \( G_0(x = 0) = 0 \).
The above derivation does more than rationalize the procedure employed by McKane and Stone. It explains why such a procedure should work for invariant operators, and should not for others. What happens when it is applied where it should not is shown in sect. 5.

The same procedure can be applied in momentum space as well, though it is slightly more delicate. The problem here manifests itself in the fact that below two dimensions, where we regularize the theory, \( p^{-2} \) is not a well-defined distribution. This makes the situation quite different from that near \( d = 4 \). The fact that, in expectation values of invariant operators, all propagators are subtracted, ensures that all propagators inside graphs become distributions, i.e., have well-defined integrals. As distributions they are defined for \( d < 2 \), and are simply analytically continued to \( d > 2 \). Each term in the perturbation expansion also becomes a distribution, which can be analytically continued.

The subtraction of a constant from the free \( \pi \) propagator implies in momentum space:

\[
G_0(p) = (p^2 + \mu^2)^{-1} \rightarrow D(p) = (p^2 + \mu^2)^{-1} - \int dq (q^2 + \mu^2)^{-1} \delta(p).
\]

(2.6)

The point now is that \( D(p, \mu) \) has a limit for \( \mu \to 0 \), which is a distribution, for both \( d < 2 \) and \( d > 2 \). This limit is just the principal part (PP) (partie finie of ref. [11]).

We show this as follows: if \( \phi(p) \) is a test function, then

\[
\lim_{\mu \to 0} \left[ (p^2 + \mu^2)^{-1} - \int dq (q^2 + \mu^2)^{-1} \delta(p), \phi \right] = \int \frac{\phi(p) - \phi(0)}{p^2} \, dp
\]

(2.7)

\[
= \lim_{\eta \to 0} \left[ \int_{|p| > \eta} \frac{\phi(p)}{p^2} \, dp - \phi(0) \int_{|p| > \eta} \frac{dp}{p^2} \right]
\]

\[
= \lim_{\eta \to 0} \left[ \int_{|p| > \eta} \frac{\phi(p)}{p^2} \, dp + \phi(0) \frac{S\eta^{d-2}}{\epsilon} \right] \equiv (PP\mu^{-2}, \phi).
\]

(2.8)

The integration is over a \( d \)-dimensional volume.

The principal part of \( p^{-(2-\epsilon)} \), \( PP\mu^{-(2-\epsilon)} \), in \( 2 + \epsilon \) dimensions is defined analogously [11]:

\[
\left( PP \frac{1}{p^{2-\epsilon}}, \phi \right) = \lim_{\eta \to 0} \left[ \int_{|p| > \eta} \frac{\phi(p)}{p^{2-\epsilon}} \, dp + \frac{\phi(0)\eta^{(n+1)\epsilon}}{(n+1)\epsilon} \right].
\]

(2.9)
The distribution $PPp^{-2}$ has a pole at $d = 2$. To see this one notes that eq. (2.7) can be written below two dimensions, as

$$
(PPp^{-2}, \phi) = \int_{|p|<1} \frac{[\phi(p) - \phi(0)]}{p^2} \, dp + \int_{|p|>1} \frac{[\phi(p) - \phi(0)]}{p^2} \, dp - \phi(0) \int_{|p|>1} \frac{dp}{p^2}
$$

$$
= \left( \left[ \frac{1}{p^2} \right], \phi \right) + \frac{S}{\epsilon} (\delta, \phi), \tag{2.10}
$$

where

$$
\left( \left[ \frac{1}{p^2} \right], \phi \right) \equiv \int_{|p|<1} \frac{[\phi(p) - \phi(0)]}{p^2} \, dp + \int_{|p|>1} \frac{\phi(p)}{p^2} \, dp \tag{2.11}
$$

is an analytic function of $\epsilon$. The pole in $\epsilon$ simply reflects the fact that $\int_{|p|>1} p^{-2} \, dp$ is divergent in the ultraviolet.

The analytic continuation of $PPp^{-2}$ to $d > 2$, is just $p^{-2}$, whose integrals over test functions are well-defined. This implies, in complete analogy with the discussion concerning coordinate space, that the correct continuation of expectation values of invariant operators to $d > 2$ is obtained by using $p^{-2}$ as a propagator and setting

$$
\int q^{-2} \, dq = 0. \tag{2.12}
$$

### 3. The logic of renormalization in coordinate and in momentum space

To underline the advantages of the proposed technique, as well as the implications of the previous section, we describe the logic of the process of renormalization, in both coordinate and momentum space.

#### 3.1. COORDINATE SPACE

In both models to be discussed below, the two-point, invariant function has the following type of perturbation expansion-regularized at $d < 2, \mu$ set to zero, and continued analytically to $d(= 2 + \epsilon) > 2$:

$$
G(x, T) = a_0 + a_1 T x^{-\epsilon} + a_2 T^2 x^{-2\epsilon} + a_3 T^3 x^{-3\epsilon} + \ldots. \tag{3.1}
$$

The functions

$$
a_i = a_i(\epsilon) \sim \epsilon^{-i}. \tag{3.2}
$$

Renormalization is performed by writing $T = i Z_1 \kappa^{-\epsilon}$, with an arbitrary mass scale.
\( G_R(x, t) = Z^{-1}G(x, tZ_1\kappa^{-t}) \),

(3.3)

with \(Z\) and \(Z_1\) chosen as functions of \(t\) and \(\epsilon\), so as to remove all poles of \(\epsilon\) on the r.h.s. of (3.3), after the powers of \(x\) are expanded in \(\epsilon\).

Writing the expansions

\[ Z^{-1} = \sum_{i=0}^{\infty} u_i t^i, \quad u_0 = 1, \]

(3.4)

\[ Z_1 = \sum_{i=0}^{\infty} v_i t^i, \quad v_0 = 1, \]

(3.5)

one can easily convince oneself that if \(G\) is computed to order \(n\) in \(T\), then elimination of the poles in the constant \((x^0)\) will determine \(u_n\) in terms of \(u_i\) and \(v_i\) \((i = 1, \ldots, n-1)\). The elimination of the poles in the coefficient of \(\ln x\) determines \(v_{n-1}\) in terms of \(u_i\) \((i = 1, \ldots, n-1)\) and \(v_i\) \((i = 1, \ldots, n-2)\).

As was already noticed by Elitzur [2], and emphasized by McKane and Stone [7], the integrals entering into the computation of \(a_n\) are those needed at one lower order in previous procedures [8, 3]. Thus \(Z\) can be obtained at one higher order with previously invested labor only, and if by some means one can find \(Z_1\), \(cr\) \(\beta\), to the next order, the exponents can be derived to the next order in \(\epsilon\) (see sect. 7).

3.2. MOMENTUM SPACE

Here the situation is again slightly more subtle. In the limit of vanishing IR cutoff the expansion of the invariant two-point function will have the general form

\[ G(p, T) = b_0\delta(p) + b_1 TPP^{-2} + b_2 T^2PP^{-2-\epsilon} + b_3 T^3PP^{-2-2\epsilon} + \ldots, \]

(3.6)

with

\[ b_i = b_i(\epsilon) \sim \epsilon^{-i+1}, \quad b_0 \sim \epsilon^0. \]

(3.7)

Using a generalization of (2.10) we expand

\[ PP^{\frac{1}{p^2-n^2}} = \left[ \frac{1}{p^2-n^2} \right] + \frac{S_0(\delta(p))}{(n+1)\epsilon} + \sum_{j=0}^{\infty} \frac{(n\epsilon)^j}{j!} \left[ \frac{\ln^j p}{p^2} \right] \]

(3.8)

with

\[ \left[ \frac{\ln^j p}{p^2} \right] \phi = \int_{|p| < 1} dp \frac{\ln(p)^j}{p^2} \left[ \phi(p) - \phi(0) \right] + \int_{|p| > 1} \phi(p) \frac{\ln(p)^j}{p^2} dp. \]
Since renormalization consists of the elimination of poles in $\epsilon$, one can regard renormalization in two ways.

(a) Ignore the $\delta$ functions altogether. This is equivalent to a restriction to test functions which vanish at $p = 0$. On this space, $[p^{-2} \ln p]$ equals $p^{-2} \ln p$ in the usual sense. The coefficients, $u_i$ and $v_i$ in (3.4), (3.5) are determined by eliminating the poles in front of $p^{-2}$ and $p^{-2}(\ln p)^2$. There is no gain in the computation relative to other techniques.

(b) If we demand, in addition, the elimination of the poles multiplying the $\delta$-function, $G_\delta(p, t)$ will be a distribution with finite limit as $\epsilon > 0$ over all test functions. This extension allows the determination of $Z$ at one higher rung in the computation ladder, as will be exemplified in sect. 4.

4. The $O(n)$ sigma model

We recapitulate briefly the application to the non-linear $\sigma$-model, arriving at the same results as McKane and Stone [7]. We do this in order to emphasize the following aspects:

(a) the procedure in momentum space;
(b) the central role of the analytic continuation in the computation of graphs;
(c) the exponentiation of the series at the critical point.

The model is defined as usual by the fields $\pi_i (i = 1, \ldots, n - 1)$ and $\sigma = (1 - \pi^2)^{1/2}$. The action is

$$A = \frac{1}{2T} \int \left\{ (\nabla \pi)^2 + \left[ \nabla (1 - \pi^2)^{1/2} \right]^2 \right\}, \quad (4.1)$$

and the statistical weight is $\exp(-A)$. We ignore the term coming from the invariant measure. It is proportional, at $d = 2$, to an integral with a positive dimension of momentum, which vanishes in dimensional regularization [3, 12].

The above rule is appended by the additional rule, sect. 2, according to which $\int k^{-2} dk = 0$. Then, if one considers the invariant two-point function

$$G(p) = \langle \pi(p) \cdot \pi(-p) \rangle + \langle \sigma(p) \sigma(-p) \rangle, \quad (4.2)$$

Fig. 1. Graphs contributing to $G(p)$ up to $O(T^3)$. Solid lines are $\pi$ propagators $p^{-2}$, broken lines are interactions equal to the square of the momentum propagating through them, wavy lines are $\sigma^2$ insertions. The external solid lines should be considered as closed isoloops. Every solid line brings a factor $T$, every broken line a factor $T^{-1}$. 

the only graphs which survive to $O(T^3)$ are shown in fig. 1. The expressions, as integrals in momentum space, are:

$$
G(p) = (2\pi)^d \delta(p) + T(n-1)p^{-2} + \frac{1}{2} T^2(n-1) \int \frac{1}{q^2(p-q)^2} \, dq
$$

$$
+ T^3(n-1) \left[ \frac{1}{2} (n-1) p^{-4} \int \frac{(p+q)^4}{q^2 q_1^2 (p+q+q_1)^2} \, dq \, dq_1 \right.
$$

$$
+ p^{-4} \int \frac{(p+q)^2 (p+q_1)^2}{q^2 q_1^2 (p+q+q_1)^2} \, dq \, dq_1
$$

$$
- \frac{1}{4} (n-1) p^2 I_2(p) - \frac{1}{2} \int \frac{(q-q_1)^2}{q^2(p+q)^2 q_1^2 (p+q+q_1)^2} \, dq \, dq_1 \right]
$$

$$
= (2\pi)^d \delta(p) + T(n-1)p^{-2} + \frac{1}{2} T^2(n-1)I_b + T^3(n-1)
$$

$$
\times \left[ \frac{1}{2} (n-1) I_c + I_d - \frac{1}{4} (n-1) I_e - \frac{1}{2} I_f \right], \quad (4.3)
$$

where $I(p)$ is the integral appearing in the $T^2$ term, and $I_e = I^2$. For each variable $q$, an integration measure $d^d q/(2\pi)^d$ is implied. The expression for $I(p)$ is

$$
I(p) = \frac{2}{\epsilon} S_d \frac{\Gamma(1+\frac{1}{2}\epsilon) \Gamma(1-\frac{1}{2}\epsilon)}{\Gamma(1+\epsilon)} p^{-(2-\epsilon)}. \quad (4.4)
$$

The integral is computed directly at $d > 2$, and the $\epsilon^{-1}$ reflects its UV divergence. It should be remembered that what is actually computed is the same integral at $d < 2$, but with IR subtracted propagators, which is then continued to $d > 2$. This integral has no IR problems even at $d < 2$.

The full expressions for the two-loop integrals appearing in (4.3) are listed in appendix A [eqs. (A.1)–(A.3)], and their $\epsilon$-expansions [eqs. (A.10)–(A.12)].

Corresponding to (4.3), we have in coordinate space

$$
G(x) = 1 + T(n-1)G_0(x) + \frac{1}{2} T^2(n-1)G_0(x) + T^3(n-1)
$$

$$
\times \left[ \frac{1}{2} (n-1) I_c(x) + I_d(x) - \frac{1}{4} (n-1) I_e(x) - \frac{1}{2} I_f(x) \right], \quad (4.5)
$$

with $I_c(x)$ through $I_f(x)$ given by (A.7)–(A.9), and their $\epsilon$-expansions, by (A.13)–
When the substitutions are all done, one finds
\[ G(x) = 1 + T(n-1)G_0(x) + ¼T^2(n-1)G_0^2(x) + T^3(n-1) \]
\[ \times \left\{ -\frac{1}{8}(n-3) - \frac{1}{8}(n-2)\varepsilon + \frac{1}{8}(n-2)\varepsilon^2 - \frac{1}{8}(n-2)\left[ \frac{3}{2}(3) - \frac{3}{2} \right] \right\} G_0^2(x), \]
(4.6)

which was previously derived in ref. [7].

When the $x^{-\eta}$ in $G_0(x)$ are expanded in powers of $\varepsilon \ln x$, and the renormalized quantities $G_R$ and $t$ are defined via
\[ G_R(x,t) = Z(t)G(x,T=t_k^{-1}), \]
with $Z(t,\varepsilon)$ and $Z_1(t,\varepsilon)$ chosen to eliminate all $\varepsilon$-poles on the r.h.s. of (4.7), one obtains $Z$ to $O(t^3)$ and $Z_1$ to $O(t^2)$ [7]:
\[ Z_1 = 1 + (n-2)\varepsilon^{-1}t + \left( \frac{n}{2} - 1 \right)\varepsilon^{-2} + \frac{1}{2}(n-2)\varepsilon^{-1} \]
\[ Z^{-1} = 1 - (n-1)\varepsilon^{-1}t + \frac{1}{2}(n-1)\varepsilon^{-2}t^2 + \frac{1}{2}(n-1) \]
\[ \times \left[ (n-3)\varepsilon^{-3} - 2(n-2)\varepsilon^{-2} - \frac{1}{2}(n-2)\varepsilon^{-1} \right]t^3. \]
(4.9)

Note that here $t$ has been defined to absorb the angular coefficient $(2\pi)^d S_d^{-1}$ of $G_0(x)$, (2.3), rather than $S_d$, which was absorbed in $t$ in the momentum representation.

On the other hand, when (A.10)-(A.12) are inserted in (4.3) one has:
\[ G(p) = (2\pi)^d S_d \delta(p) + T(n-1)p^{-2} + T^2(n-1)\varepsilon^{-1}p^{-(2-\varepsilon)} - \frac{1}{2} T^3(n-1) \]
\[ \times \left[ (n-3)\varepsilon^{-2} + (n-2)\varepsilon^{-1} - \frac{3}{2}(n-2) \right]p^{-(2-\varepsilon)}. \]
(4.10)

In this equation $G(p)$ was multiplied by $S_d$, $T$ absorbed an $S_d$, and $A_d$ is given by (2.3). The powers of $p$ stand for their principal parts.

The form (4.6) is an answer to one of the concerns motivating the present work. Namely, it leads to an explicit direct exponentiation of the renormalized perturbation series, at the critical point, to a power of the distance. Namely, the renormalization constants (4.8) and (4.9) lead to
\[ \beta(t) = et - (n-2)t^2 - (n-2)t^3, \]
(4.11)
\[ \xi(t) = \beta(t)(\partial \ln Z/\partial t) = (n-1)t + \frac{1}{2}(n-1)(n-2)t^2. \]
(4.12)

[Hikami and Brezin [8] have computed the next $-O(t^4)$ term in $\beta(t).]
From (4.11) one has for the fixed point—the critical temperature—

\[ t_c = \frac{\epsilon}{(n-2)} - \frac{\epsilon^2}{(n-2)^2} + \frac{\epsilon^3}{6(n-2)}. \]  

(4.13)

The correlation function should behave, at \( t_c \), as

\[ G(x) \sim C x^{-(\eta+\epsilon)}, \]  

(4.14)

where [3]

\[ \eta + \epsilon = \xi(t_c). \]  

(4.15)

To order \( \epsilon^3 \), the renormalized correlation function, eq. (4.7), \( G_R(x) \) has the form

\[
G_R(x) = 1 - (n-1)(n-2) \left[ \frac{3}{4} - \frac{1}{2} \xi(3) \right] x^2 - \left[ (n-1)x + \frac{3}{4}(n-1)(n-2)x^3 \right] \\
\times \ln \kappa x + \frac{1}{2} (n-1) \left[ t^2 + \epsilon t - (n-2)x^3 \right] \ln^2 \kappa x \\
+ (n-1) \left[ 1 - \frac{1}{4} \epsilon x^2 - \frac{1}{2} \epsilon t^2 + \frac{3}{4}(n-3)x^3 \right] \ln^3 \kappa x .
\]  

(4.16)

Substituting \( t_c \), from (4.13), one finds

\[
G^c_R(x) = 1 - \frac{1}{8} \frac{n-1}{(n-2)^2} \left[ 3 - 2\xi(3) \right] x^2 - \frac{n-1}{n-2} \epsilon \left[ 1 - \frac{\epsilon}{n-2} + \frac{\epsilon t^2}{2(n-2)^2} \right] \ln \kappa x \\
+ \frac{1}{2} \frac{(n-1)^2}{n-2} \epsilon^2 \left[ 1 - \frac{2\epsilon}{n-2} \right] \ln^2 \kappa x - \frac{n-1}{6(n-2)} \epsilon^3 \ln^3 \kappa x ,
\]  

(4.17)

which leads to

\[ C = 1 - \frac{1}{8} \frac{n-1}{(n-2)^2} \left[ 3 - 2\xi(3) \right] \epsilon^3 , \]  

(4.18)

\[ \eta = \frac{\epsilon}{(n-2)} - \frac{(n-1)}{(n-2)^2} \epsilon^2 + \frac{n(n-1)\epsilon^3}{2(n-2)^2} , \]  

(4.19)

in expected agreement with previous results [38].

Finally, a comment on the 'exponentiation' in momentum space. The function \( G(p) \), (4.10), is renormalized by the same constants \( Z_1 \) and \( Z_2 \), of eqs. (4.8) and (4.9). The renormalization here is not standard in the sense indicated in sect. 3. Namely, the removal of \( \epsilon \)-poles has to include the poles proportional to \( \delta \)-functions. In fact, that is precisely what the multiplication of the \( \delta \)-function in (4.10) by \( Z^{-1} \) does.
The renormalized function, after setting \( \kappa = 1 \), is given by

\[
G_R(p) = (2\pi)^d S_d \delta(p) + (n - 1) \epsilon \left[ p^{-2} \right] \\
+ (n - 1) \epsilon^2 \left[ \epsilon^{-1} \left[ p^{-2+\epsilon} \right] - \epsilon^{-2} \left[ p^{-2} \right] + \frac{1}{4} \epsilon^3 (3) p^{-2+\epsilon} \right] \\
+ (n - 1) \epsilon^2 \left[ \frac{1}{2} \epsilon^{-1} (n - 2) - \frac{1}{2} \epsilon^{-2} (n - 3) \right] \left[ p^{-2} \right] \\
+ (n - 3) \epsilon^{-2} \left[ p^{-2+\epsilon} \right] + \frac{3}{4} (n - 3) \epsilon (3) p^{-2+\epsilon} \\
+ \frac{1}{6} \epsilon \left[ \frac{1}{6} (n + 6) \epsilon (3) - \frac{2}{3} (n - 2) \right] p^{-2+2\epsilon} \right].
\]

(4.20)

The notation \( [p^m] \) stands for the \( PP(p^m) \) - (pole part) in analogy with (2.11), namely,

\[
[p^{-2+\epsilon}] = PP(p^{-2+\epsilon}) - \frac{(2\pi)^d S_d}{(j + 1) \epsilon} \delta(p).
\]

(4.21)

Using expansion (3.8) we arrive at

\[
G_R(p) = (2\pi)^d S_d \delta(p) \left\{ 1 + \frac{n - 1}{2(n - 2)^3} \left[ -\frac{1}{4} (n - 2) + \frac{1}{12} \epsilon (3) (7n - 12) \right] \right\} \\
+ (n - 1) \left[ \epsilon / (n - 2) - \epsilon^2 / (n - 2)^2 + \epsilon^3 n / (2(n - 2)^3) \right] \left[ p^{-2} \right] \\
+ (n - 1) \epsilon^2 \left[ 1 / (n - 2)^2 - \epsilon n / (n - 2)^3 \right] \left[ p^{-2} \ln p \right] \\
+ \frac{1}{2} (n - 1) \epsilon^3 \left[ 1 / (n - 2)^3 \right] \left[ p^{-2} \ln^2 p \right] + O(\epsilon^4).
\]

(4.23)

This should be compared with the expansion:

\[
C'(\epsilon) \left[ p^{2-\eta} \right] = C'(\epsilon) \left\{ \frac{(2\pi)^d S_d \delta(p)}{\eta + \epsilon} + \left[ p^{-2} \right] + \eta \left[ p^{-2} \ln p \right] \ldots \right\},
\]

(4.24)

from which one can read \( C', \eta + \epsilon \) and \( \eta \), as well as verify exponentiation.

5. IR finiteness and renormalization: a conjecture

The procedure outlined so far raises the following question: if the prescription-analytic continuation of \( d > 2 \), of a dimensionally regularized quantity is applied to a Green function which is not invariant, will there be a sign?
In other words, if any Green function, $\Gamma^{(2)}_{\text{av}}$, for example, is computed by a direct calculation in $d > 2$, setting to zero $G(0)$, as well as derivatives of $G(x)$ at $x = 0$, there will be no IR problems left. Would anything indicate that something is wrong?

We do not have a full answer to this question. But on the basis of examples it is conjectured that a sign will show.

Consider the two-point vertex $\Gamma^{(2)}_{\text{av}}(p)$. It was written explicitly in ref. [3], eq. (65), to the order of two loops. The rules are that $H \rightarrow 0$, and the momentum integrals of $p^a$, for $a > -2$, are set to zero. This eliminates all the one-loop terms, as well as most of the two-loop ones. The only survivors are graphs c and d of fig. 1. The result is

$$
\Gamma^{(2)}_{\text{av}}(p) = T^{-1}p^2 - \frac{1}{2}(n-1)T \int \frac{(q + q_i)^2 dq dq_i}{q^2 q_i^2 (p + q + q_i)^2} - T \int \frac{(p + q)^2(p + q_i)^2 dq dq_i}{q^2 q_i^2 (p + q + q_i)^2}
$$

$$
= T^{-1}p^2 - T \left[ \frac{1}{2}(n-1)(1 + \frac{1}{2} \epsilon) \right] \frac{\Gamma^5(1 + \frac{1}{2} \epsilon)(1 - \epsilon)}{\Gamma(2 + \frac{1}{2} \epsilon)} \frac{p^{2+2\epsilon}}{\epsilon^2}.
$$

The two terms in the square brackets in (5.2) correspond to the two integrals in (5.1) (see appendix B) apart from a factor $S_2^2$, given by (2.4), which is absorbed in $T^2$, to conform with the notation of Brézin and Zinn-Justin [3].

Expanding (5.2) in powers of $\epsilon$, one has

$$
\Gamma^{(2)}_{\text{av}}(p) = T^{-1}p^2 - \frac{1}{2}T p^2 (\epsilon^{-2} + 2 \epsilon^{-1} \ln p) \left[ (n-1) + \frac{1}{2} (n+1) \epsilon \right].
$$

This function is not renormalizable! No choice of $Z$ and $Z_1$ will eliminate the term proportional to $\epsilon^{-1} p^2 \ln p$. The same thing happens to the $\sigma\sigma$ propagator.

This leads us to conjecture that if the procedure, outlined in sects. 2–4, is applied mechanically to a function that is not finite for $d < 2$, as $\mu \rightarrow 0$, the result will be an unrenormalizable function. The impossibility of removing the $\epsilon$-poles is the desired signature.

6. The application to the $\text{CP}^{(n-1)}$ model

The $\text{CP}^{(n-1)}$ model in $d$ dimensions is described by [6]

$$
A = \frac{1}{2T} \int \left[ \tilde{\partial}_\mu z^* \partial_\mu z^* + \frac{1}{4} \left( z^* \tilde{\partial}_\mu z^* \right)^2 \right] d^d x,
$$

where

$$
z_i \tilde{\partial}_\mu z^* = z^*_i \tilde{\partial}_\mu z_i - z_i \partial_\mu z^*_i,
$$

$$
z^*_i z_i = 1.
$$

The coefficient $T$ plays the same role as in the $O(n)$ $\sigma$-model [9].
The generating functional for the euclidean Green functions [6] is given by

\[ Z[j, j^*] = \int Dz Dz^* \delta(|z(x)|^2 - 1) e^{-\int dz \cdot z^* \cdot z} e^{\int j(x) \cdot z^* \cdot j^*}. \] (6.4)

The model has, besides the global U(n) symmetry, a local gauge invariance, i.e.,
the action is invariant under a transformation

\[ z_j \rightarrow e^{i\phi(x)}z_j, \quad z^*_j \rightarrow e^{-i\phi(x)}z^*_j, \] (6.5)

for any real function \( \phi(x) \).

In order to benefit from Elitzur's theorem one has to look for fully invariant Green functions. Moreover, the renormalization will be multiplicative [12] if the invariant is constructed from an irreducible representation of SU(n). This we do as follows: let \( \lambda^a (a = 1 \ldots n^2 - 1) \) be a basis for the Lie algebra of SU(n). The fields

\[ \phi^a \equiv z^* \lambda^a z \]

transform according to the adjoint representation of SU(n), and

\[ G(x) = \left< \sum_a \phi^a(x) \phi^a(0) \right> \]

is the desired invariant two-point function. This two-point function is the simplest gauge-invariant SU(n) symmetric propagator one can construct. With the standard normalization

\[ Tr \lambda^a \lambda^b = 2 \delta_{ab}, \] (6.6)

\[ G(x) \] can be rewritten, in terms of the constituent fields \( z \), as

\[ G(x) = 2 \left< z^*(x) \cdot z(0) z(x) \cdot z^*(0) \right> - \frac{1}{n}. \] (6.7)

For \( n = 2 \), \( \lambda^a \) are the Pauli matrices, the action becomes \( (1/8T)(\partial_\mu \phi^a)^2 \), which is the O(3) \( \sigma \)-model, with a coupling constant (temperature) \( 4T \). The invariant propagator eq. (6.7), becomes \( (\sum_a \phi^a(x) \phi^a(0)) \), which provides a useful check for the calculation.

In order to perform the calculation, the gauge has to be fixed. In what follows a "unitary" gauge, specified by

\[ \text{Im} z_n = 0 \] (6.8)

will be employed. In addition, we parametrize the first \( n - 1 \), complex \( z \)'s by \( 2n - 2 \)
independent real fields:
\[ z_k = a_k + ib_k, \quad k = 1, \ldots, n - 1, \quad (6.9) \]
\[ z_n = \left[ 1 - \sum_a (a_a^2 + b_a^2) \right]^{1/2}. \quad (6.10) \]

Next we rescale the field by a factor $\sqrt{T}$ to bring the action to the form:
\[ A = \frac{1}{2} \int \left[ (\partial_\mu a_k)^2 + (\partial_\mu b_k)^2 \right] + \frac{1}{2T} \int \left\{ \partial_\mu \left[ 1 - T(a^2 + b^2) \right]^{1/2} \right\}^2 \]
\[ - \frac{1}{2} T \int (a_\mu \partial_\mu b_k - b_\mu \partial_\mu a_k)^2, \quad (6.11) \]

where the integrals, here and in sect. 7, are over $d$-dimensional coordinate space.

7. Calculation to third order

The action, expanded to second order in $T$, reads
\[ A = \frac{1}{2} \int \left[ (\partial_\mu a)^2 + (\partial_\mu b)^2 \right] + \frac{1}{2} T \int (a \partial_\mu a + b \partial_\mu b)^2 - \frac{1}{2} T \int (a \partial_\mu b - b \partial_\mu a)^2 \]
\[ + \frac{1}{2} T^2 \int (a^2 + b^2) (a \partial_\mu a + b \partial_\mu b)^2 \quad (7.1) \]

We expand the invariant propagator (6.7) to third order in $T$:
\[ G(x) = 2(n - 1)/n + 4T \left[ \langle v(x) \cdot v(0) \rangle^{(0)} - \langle v^2(0) \rangle^{(0)} \right] \]
\[ + 2T^2 \left[ \langle (v(x) \cdot v(0))^2 \rangle^{(0)} + \langle (a(x) \cdot b(0) - b(x) \cdot a(0))^2 \rangle^{(0)} \right] \]
\[ + \langle v^2(x) v^2(0) \rangle^{(0)} - 2 \langle v^2(0) v(x) \cdot v(0) \rangle^{(0)} + 2 \langle v(x) \cdot v(0) \rangle - 2 \langle v^2(0) \rangle^{(1)} \]
\[ + 2T^3 \left[ \frac{1}{2} \langle v^2(x) v^2(0) v(x) \cdot v(0) \rangle^{(0)} - \frac{1}{2} \langle v(x) \cdot v(0) v^2(0) \rangle^{(0)} \right] \]
\[ + 2 \langle v(x) \cdot v(0) \rangle^{(2)} - 2 \langle v^2(0) \rangle^{(2)} + \langle v^2(x) v^2(0) \rangle^{(1)} + \langle (v(x) \cdot v(0))^2 \rangle^{(1)} \]
\[ - 2 \langle v^2(0) v(x) \cdot v(0) \rangle^{(1)} + \langle (a(x) \cdot b(0) - b(x) \cdot a(0))^2 \rangle^{(1)} \right] \quad (7.2) \]

where we set
\[ v \equiv (a, b) \]

and $\langle f \rangle^{(n)}$ denotes the average of the operator $f$ at order $n$ in $T$. 
(a) First-order calculation. As explained in sect. 2, above two dimensions

\[ \langle v^2(0) \rangle^{(0)} = (2n - 2)G_0(0) = 0. \]

Hence

\[ G_1(x) = 2(n - 1)/n + 4T(2n - 2)G_0(x) + O(T^2), \]

with

\[ G_0(x) = [(2\pi)^dS_d \delta x^d]^{-1} = \langle v(x) \cdot v(0) \rangle^{(0)}/(2n - 2), \]

as in the O(n) \( \sigma \)-model.

(b) Second-order calculation. The rule \( G_0(x = 0) = 0 \) allows us to obtain the second-order result without carrying out any integrations

\[ G_2(x) = 2(n - 1)/n + 4T(2n - 2)G_0(x) + 2T^2 \left[ \langle (v(x) \cdot v(0))^2 \rangle^{(0)} + \langle v^2(x) v^2(0) \rangle^{(0)} - 2 \langle a(x) \cdot b(0) b(x) \cdot a(0) \rangle^{(0)} \right]. \tag{7.3} \]

(c) Third-order calculation. The interaction part of the action (7.1) is composed of three terms. The first term

\[ A_1 = \frac{1}{2} T \int (a^2 + b^2)^2 \tag{7.4} \]

is just the interaction term in a O(2n - 1) non-linear \( \sigma \)-model.

The second term

\[ A_2 = -\frac{1}{2} T \int (a \partial \mu a - b \partial \mu b)^2 \tag{7.5} \]

introduces new vertices which are shown in fig. 2, together with the vertices of the non-linear \( \sigma \)-model. The convention is: a single line stands for an \( a \)-line, a double line for a \( b \)-line, and a stroke for a derivative. The third term

\[ \frac{1}{2} T^2 \int (a^2 + b^2)(a \partial \mu a + b \partial \mu b)^2 \]

does not contribute to (7.2) because of the rule \( G_0(0) = \nabla G_0(0) = 0. \)

Fig. 2. Vertices of the CP\((n-1)\) interaction. The first three diagrams are generated by (7.4), the last three by (7.5).
The diagrams entering the calculation of $\langle \psi^2(x) \psi(0) \rangle^{(1)}$, in (7.2) are shown in fig. 3. The faithful representation of the interactions is suppressed. Their sum is

$$\langle \psi^2(x) \psi(0) \rangle^{(1)} = -16(n - 1) n J_a$$  \hspace{1cm} (7.6)$$

The free propagator of the $a$, and $b$ fields are identical, therefore, the presence of single and double lines in a diagram only influences it combinatorical weight. In the following we will represent all the free propagators by a single line, in order to simplify the notation and reduce the number of graphs.

The calculation of $\langle (a(x) \cdot b(0) - b(x) \cdot a(0))^2 \rangle^{(1)}$ and $\langle (\psi(x) \cdot \psi(0))^2 \rangle^{(1)}$, for example, will be expressed in terms of diagrams of fig. 4. The result is

$$\langle [a(x) \cdot b(0) - b(x) \cdot a(0)]^2 \rangle = 8n(n - 1)(J_a - J_b).$$  \hspace{1cm} (7.8)$$

Using (7.6)–(7.8), we obtain (see appendix B)

$$2\left[ \langle \psi^2(x) \psi(0) \rangle^{(1)} + \langle (\psi(x) \cdot \psi(0))^2 \rangle^{(1)} + \langle (a(x) \cdot b(0) - b(x) \cdot a(0))^2 \rangle^{(1)} \right] = -\frac{64}{3} n(n - 1) \left[ 1 - \frac{1}{2} e^3 \lambda \left( 3 \right) \right] G_0^3(x).$$  \hspace{1cm} (7.9)$$

Next we express $4\langle \psi(x) \cdot \psi(0) \rangle^{(2)}$, in (7.2), in terms of the integrals corresponding to fig. 5:

$$\langle \psi(x) \cdot \psi(0) \rangle^{(2)} = 32n(n - 1) \left[ 2 K_a + K_a + K_d \right] = \frac{4}{3} n(n - 1) \left( \frac{6 + \epsilon}{1 + \frac{3}{2} \epsilon} \right) G_0^3(x).$$  \hspace{1cm} (7.10)$$

(see appendix B.)
The sum of the coefficients of $K_c, K_\theta, K_\beta_1$, entering (7.10), turns out to be zero. The term $-4\langle v^2(0)v(x)\cdot v(0)\rangle^{(1)}$ in (7.2) is calculated in terms of the diagrams listed in fig. 6 and evaluated in appendix B. The result is

$$-4\langle v^2(0)v(x)\cdot v(0)\rangle^{(1)} = 32n(n - 1)\left[L_a + L_b\right] = \frac{16}{3} n(n - 1)G_0^3(x). \quad (7.11)$$

Finally, the zeroth-order average in the brackets multiplying $T^3$ in (7.2) is

$$\langle v^2(x)v^2(0)v(x)\cdot v(0)\rangle^{(0)} = 8(n - 1)nG_0^3(x), \quad (7.12)$$

while $-\langle v(x)\cdot v(0)v^4(0)\rangle^{(0)} - 4\langle v^2(0)\rangle^{(2)} = 0$ because $G_0(0) = \nabla G_0(0) = 0$. Assembling (7.9)–(7.12), we arrive at

$$G_3(x) = 2(n - 1)/n + 2(n - 1)T G_0(x) + \frac{1}{2} n(n - 1)T^2 G_0^2(x)$$

$$-\frac{1}{6}(n - 1)nT^3 G_0^3(x)\left\{1 - \frac{1}{2}e^2 + \frac{2}{3} e^3 - \frac{5}{6} e^4 + \frac{2}{3} e^5 - \frac{5}{6} e^6\right\}, \quad (7.13)$$

where $T$ was replaced by $4T$ throughout.

*(d) Renormalization.* Demanding that

$$G_R(x, t) = Z^{-1}G(x, Z_1 t \kappa^{-1})$$

have no poles in $\epsilon$, order by order in $\epsilon$, we obtain

$$Z = 1 + \frac{n}{\epsilon} t + \frac{3n^2}{4\epsilon^2} t^2 + t^3 \left(\frac{n^3}{4\epsilon^3} + \frac{n^2}{6\epsilon^2} + \frac{1}{8}\frac{n^2}{\epsilon}\right), \quad (7.14)$$

$$Z_1 = 1 + \frac{n}{2\epsilon} t + \left(\frac{n^2}{4\epsilon^2} + \frac{n}{4\epsilon}\right) t^2, \quad (7.15)$$

where $t$, the renormalized temperature is given by

$$Z_1 t = T \kappa^t.$$
(e) **Comparison with previous results.** Using a magnetic field as an infrared regulator, Hikami [9] calculated the Green function

$$\langle v_1(x)\sigma(x)v_1(0)\sigma(0)\rangle,$$

where \(\sigma = \sqrt{1 - \alpha^2}\), and obtained \(Z\) and \(Z_1\) to second order in \(\alpha\). Since the composite operator calculated by Hikami belongs to the same irreducible representation of SU\((n)\) as the operator used in this work, renormalization constants agree, to the order they were calculated.

(\(f\)) **Critical exponents.** The two-point function we calculated is, by construction, gauge invariant. Therefore, its divergence at the critical point is characterized by a gauge-independent exponent \(\eta\):

$$G_R(k) \propto k^{-(2-\eta)}, \quad t = t_c.$$  \hspace{1cm} (7.16)

Our calculation yields, in addition to Hikami's results, the coefficient of \(t^3\) in \(Z\). This allows us to calculate \(\eta\) to third order in \(\epsilon\), given the fixed point at \(O(\epsilon^3)\), or \(\beta(t)\) to fourth order in \(\epsilon\). This information is not provided by our calculation. Hikami [9] derived the coefficient of \(t^4\) in \(\beta(t)\) using information from the \(1/n\) expansion, the \(O(3)\) non-linear \(\sigma\)-model (CP\(^{1(n)}\)), and assuming strict scale invariance of CP\(^{(-1)}\) at two dimensions. (The justification of the last assumption is still missing.)

His result is

$$\beta(t) = \epsilon t - \frac{1}{2}nt^2 - \frac{1}{4}nt^3 - \left(\frac{3}{16}n^2 + \frac{1}{4}n\right)t^4,$$  \hspace{1cm} (7.17)

from which one can compute \(\nu^{-1} = -\beta'(t_c)\) to \(O(\epsilon^3)\).

We can proceed to calculate

$$\eta = -\epsilon + \xi(t_c)$$  \hspace{1cm} (7.18)

to third order in \(\epsilon\). The result is

$$\xi(t) = \beta(t) \frac{3}{\partial T} \ln Z = nt + \frac{1}{2}n^2t^3 + O(t^4),$$  \hspace{1cm} (7.19)

$$\eta = \epsilon - \frac{4}{n}\epsilon^2 + \frac{12}{n^2}\epsilon^3 + O(\epsilon^4).$$  \hspace{1cm} (7.20)

8. **Conclusion**

The advent of Elitzur's theorem has eliminated the difficulties associated with the Goldstone \(\pi\)'s which appear when one studies the perturbation expansion.
about a vacuum with a broken continuous symmetry, in a situation in which the symmetry is dynamically restored.

As was pointed out by McKane and Stone [7], the lattice regularization is very cumbersome, dimensional regularization is preferable and is a much more efficient calculational tool. Above we have systematized the dimensional approach, cleared up questions such as: (i) the precise meaning of the analytic continuation to \( d > 2 \) and its close connection to the underlying theorem; (ii) the form of the procedure in momentum space; (iii) the explicit exponentiation of the Green function at the point of scale invariance at \( d = 2 + \epsilon \). In addition, the calculations were extended to \( \mathbb{C}P^{(\alpha - 1)} \).

The whole subject may seem redundant at \( d > 2 \), since there, at \( T < T_c \), the symmetry is indeed broken. But the \( \epsilon \)-expansion is, of course, an expansion which is very sensitive to what happens at \( d = 2 \). Furthermore, one is interested in including the tip of the coexistence curve, \( T = T_c \), in the domain of the calculation. There the symmetry is restored, and the correlations decay by power laws.

The resulting, infrared-finite, perturbation theory is as useful as our ability to control the series. At \( d = 2 + \epsilon \) and \( T < T_c \), both long- and short-distance behavior are controlled by fixed points which are small (\( O(\epsilon) \)), or even asymptotically vanishing. At \( T > T_c \), or at \( d = 2 \) for \( T > 0 \), the long-distance behavior is determined by large coupling. In that case some resummation technique is needed, such as the \( 1/N \) expansion [4, 6, 14].

On this question our view differs from that expressed in ref. [7]. There is no good reason for doubting the sufficiency of the perturbation series even beyond the point of scale invariance. A finite number of terms will not give, of course, the mass (the correlation length) at \( T > T_c \). But the \( 1/N \) expansion [4, 6] for the \( O(n) \) non-linear-\( \sigma \)-model, as well as for \( \mathbb{C}P^{(\alpha - 1)} \), gives not only the masses, but a spectrum with the correct symmetry.

In fact, the generation of the mass in the \( 1/N \) expansion may provide a hint to the identification of the mass in perturbation theory.

Finally, to support the view that all is well with the perturbation expansion of invariant Green functions, we would like to take issue with some of the comments made by Jevicki [5]. Attributing the generation of the mass, in the large-\( N \) expansion, to IR divergences, Jevicki concludes that in the large-\( N \) limit the cancellations discussed above do not take place.

This conclusion is neither true nor possible. The cancellations implied by Elitzur's theorem, as well as in Jevicki's earlier treatment of the effective potential in the linear \( \sigma \)-model at \( d = 2 \), take place for any value of \( N \). Hence they must occur at every order in perturbation theory, for every power of \( N \) independently. In particular, the coefficient of the leading power in \( N \), at every order in perturbation theory, must be finite for an invariant operator.

Nevertheless, the origin of the mass in the large-\( N \) expansion [4, 6, 14] is indeed the IR divergences, but the real ones. Namely, after Elitzur's theorem rics the
theory of logarithms of a symmetry breaking mass, logarithms of the external momentum, or coordinate, take their place. These are real, and they accumulate to produce the mass.

It seems worthwhile to insist on this point because the perturbation expansion, the $\epsilon$-expansion, and the large-$N$ expansion complement each other in such a fruitful way that no effort should be spared in eliminating internal divisions.

Finally, these techniques should be applied to other situations, in which the appearance of infrared-regulating masses do not contribute to the clarity of the issues. Cases which immediately come to mind are: (i) the mixing of operators of high dimension in the non-linear $\sigma$-model [15]; and (ii) the fluctuations of surfaces, breaking euclidean invariance [16].

We are very grateful for many, very useful discussions with S. Elitzur, A. McKane, E. Rabinovici, E. Witten and R.K.P. Zia.

Appendix A

TWO-LOOP INTEGRALS IN MOMENTUM AND IN COORDINATE SPACE

We provide below a list of the two-loop integrals appearing in fig. 1, together with their $\epsilon$-expansions, the computations are by now standard [13].

In momentum space. With subscripts corresponding to the graphs in fig. 1, we have:

\begin{equation}
I_c(p) = S_d^2(e^{-2 + \frac{1}{2} \epsilon}) p^{-2 + 2\epsilon} \Gamma^2(1 + \frac{1}{2} \epsilon) \Gamma(1 - \epsilon) / \Gamma(2 + \frac{3}{2} \epsilon),
\end{equation}

\begin{equation}
I_d(p) = S_d^2(\frac{1}{2} e^{-1}) p^{-2 + 2\epsilon} \Gamma^2(1 + \frac{1}{2} \epsilon) \Gamma(1 - \epsilon) / \Gamma(2 + \frac{3}{2} \epsilon),
\end{equation}

\begin{equation}
I_e(p) = S_d^2(4 e^{-2}) p^{-2 + 2\epsilon} \Gamma^2(1 + \frac{1}{2} \epsilon) \Gamma^2(1 - \frac{1}{2} \epsilon) / \Gamma^2(1 + \epsilon) = -2 I_f,
\end{equation}

with

\begin{equation}
S_d = \left[2^{d-1} \pi^{d/2} \Gamma(\frac{d}{2})\right]^{-1}.
\end{equation}

Apart from the fact that the number of graphs is greatly reduced, the computations of the graphs is also facilitated. For example, the computation of $I_f$ proceeds as follows:

\begin{equation}
I_f = \int \frac{(q - q_1)^2 d q d q_1}{q^2(p + q)^2 q_1^2(p + q_1)^2} = -2 \left[ \int \frac{q d q}{q^2(p + q)^2} \right]^2 + 2 \frac{d q d q_1}{(p + q)^2 q_1^2(p + q_1)^2}.
\end{equation}

The second term vanishes because of (2.10). Thus the calculation reduces to

\begin{equation}
I_f = -2 \left[ \int \frac{q d q}{q^2(p + q)^2} \right]^2.
\end{equation}
To write the expressions for the same terms in coordinate space, one can go back and re-express all graphs in terms of $G_0$, eq. (2.5), and its derivatives [7]. It is more efficient to Fourier transform (A.1)–(A.3), using the simple formula

$$I_j = \int \frac{dp}{(2\pi)^d} \mathrm{e}^{ipx} p^{-2+\epsilon} = \frac{\Gamma(\frac{1}{2}(j+1)\epsilon)2^{(j+1)/2\epsilon}}{2(2\pi)^{(d/2)\epsilon}(\Gamma(1-\epsilon/2))} x^{-(j+1)\epsilon},$$

and then to re-express $x^{-\epsilon}$ in terms of $G_0(x)$, if it is so desired [7] via (2.5). This has to be done only once for all terms of the same order in $T$. Eq. (B.5) can be written as

$$I_j = \frac{\Gamma(\frac{1}{2}(j+1)\epsilon)2^{(j+1)/2\epsilon}}{2(2\pi)^{(d/2)\epsilon}(\Gamma(1-\epsilon/2))} (S_{d\epsilon})^{"} G_0^{d+1}. \quad (A.6)$$

When this is performed on (A.1)–(A.3) the result is

$$I_c(x) = \frac{1}{2} (1 + \frac{1}{2} \epsilon)(1 + \frac{3}{2} \epsilon)^{-1} G_0^0(x), \quad (A.7)$$

$$I_d(x) = \frac{1}{6} \epsilon (1 + \frac{3}{2} \epsilon)^{-1} G_0^0(x), \quad (A.8)$$

$$I_e(x) = -2I_t(x) = \frac{4}{3} \left[ \Gamma(1 + \frac{1}{3} \epsilon) \Gamma(1 + \frac{5}{3} \epsilon) \Gamma^2(1 + \frac{1}{2} \epsilon)/\Gamma^2(1 + \epsilon) \Gamma(1 - \epsilon) \right] G_0^0(x). \quad (A.9)$$

The $\epsilon$-expansions of the divergent part of the integrals are

$$I_c(p) = S_0[\epsilon^{-2} - \epsilon^{-1} + \frac{1}{2} + \epsilon(\xi(3) - \frac{3}{2})] p^{-2+2\epsilon}, \quad (A.10)$$

$$I_d(p) = S_0\left[ (\frac{1}{3} \epsilon)^{-1} - \frac{1}{2} + (\frac{7}{2} + \frac{5}{2} \xi(3)) \epsilon \right] p^{-2+2\epsilon}, \quad (A.11)$$

$$I_e(p) = -2I_t(p) = S_0^2 \left[ \epsilon^{-2} + \frac{1}{2} \xi(3) \right] p^{-2+2\epsilon}, \quad (A.12)$$

$$I_c(x) = \frac{1}{2} \left[ (1 - \epsilon + \frac{3}{2} \epsilon^2 - \frac{3}{4} \epsilon^3) \right] G_0^0(x), \quad (A.13)$$

$$I_d(x) = \frac{1}{6} \left[ \epsilon - \frac{3}{2} \epsilon^2 + \frac{5}{4} \epsilon^3 \right] G_0^0(x), \quad (A.14)$$

$$I_e(x) = \frac{4}{3} \left[ (1 - \frac{3}{2} \epsilon^2) \xi(3) \right] G_0^0(x) = -2I_t(x). \quad (A.15)$$

To these should be added the $\epsilon$-expansion of $I(p)$, eq. (4.4), namely

$$I_0(p) = I(p) = S_0(2\epsilon^{-1})(1 + \frac{1}{2} \epsilon^3 \xi(3)) p^{-2+\epsilon}, \quad (A.16)$$

$$I_0(x) = G_0^0(x). \quad (A.17)$$
In the same way we may show that \( K_t \) and \( I_d \), (A.8), are related via

\[
    I_d = -\frac{1}{3} G_0^3(x) + 4K_t
\]

(B.4)

Finally, one can express \( K_b, K_c, K_d, K_e \) in terms of \( J_a \) and \( J_t \) through integration by parts the result is

\[
    K_c = \frac{1}{3} (K_t - K_a) = G_0^3(x) \epsilon / 12(2 + 3\epsilon), \quad \text{(B.5)}
\]

\[
    K_e = -2K_c = -G_0^3(x) \epsilon / 6(2 + 3\epsilon), \quad \text{(B.6)}
\]

\[
    K_b = -(K_t + K_e) = -G_0^3(x)(1 + \epsilon) / 6(2 + 3\epsilon), \quad \text{(B.7)}
\]

\[
    K_d = -2K_b = G_0^3(x)(1 + \epsilon) / 3(2 + 3\epsilon). \quad \text{(B.8)}
\]

The integrals corresponding to the diagrams in fig. 4 are related to the integrals \( I_t, I_e \) of appendix A through integration by parts. The results are

\[
    J_b = \frac{1}{3} I_t = \frac{1}{3} \left( 1 - \frac{3}{4} \epsilon \right) G_0^3(x), \quad \text{(B.9)}
\]

\[
    J_a = \frac{1}{3} I_e = -\frac{2}{3} \left( 1 - \frac{3}{4} \epsilon \right) G_0^3(x). \quad \text{(B.10)}
\]

References

    E. Witten, Nucl. Phys. B149 (1979) 285