Transport coefficients close to the mobility edge and nonlinear $\sigma$-model composite operators

C. Castellani* and G. Kotliar
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We have expressed the conductivity and the ultrasonic attenuation coefficient in terms of nonlinear $\sigma$-model composite operators. The conductivity is represented by the sum of local and nonlocal terms. The nonlocal part is given by a correlation function of operators involving derivatives. It is expected to vanish to three-loop order in the orthogonal case but gives a nonzero result at two-loop order in the unitary case. The ultrasonic attenuation coefficient is represented by a local term only. We use this result to derive the scaling behavior of the ultrasonic attenuation coefficient close to the mobility edge in both the orthogonal and the unitary case.

I. INTRODUCTION

The understanding of the effect of disorder on the transport properties of electrons has seen much progress in recent years. In a seminal paper Anderson showed that strong disorder localizes all the electronic eigenstates in three dimensions. The transition from the metallic to the insulator regime as the strength of the disorder is increased is known as the Anderson transition. Wegner and Abrahams et al. analyzed this transition using scaling ideas borrowed from the theory of second-order phase transitions. This transition, which is believed to be continuous, has been analyzed using a variety of renormalization-group techniques. A most powerful approach to extract the critical behavior of the diffusion constant and of the different moments of the wave functions is Wegner mapping of the localization problem onto a model of interacting matrices, the nonlinear $\sigma$ model.

In this paper we use the nonlinear $\sigma$ model to study in a unified context the critical behavior of the conductivity (which is related to a current-current correlation function) and the ultrasonic attenuation (which is related to a stress-stress correlation function). The study of transport coefficients in the framework of the nonlinear $\sigma$ model is nontrivial because current and stress operators have no obvious representation in terms of the matrix variables $Q$ of the nonlinear $\sigma$ model. The critical behavior of the ensemble-averaged conductivity is well known since it is related via the Einstein relation to the diffusion constant which appears very naturally in the nonlinear $\sigma$ model.

We shall find a different representation of the current-current correlation function in terms of composite operators of the nonlinear $\sigma$ model. This representation is relevant from both a theoretical and a practical point of view since it allows the calculation of the moments of the sample-to-sample conductance fluctuation in a very direct way. A similar representation of the conductivity tensor was used by Prusken in the context of the quantized Hall effect.

Kotliar and Ramakrishnan pointed out that the ultrasonic attenuation coefficient $\tilde{\alpha}$, which is given in terms of the stress-stress correlation function, is affected by Anderson localization. They argued that the anomalously enhanced backscattering (which drives the conductivity to zero) enhances the ultrasonic attenuation coefficient, and verified this assertion using the $1/k_F l$ expansion, valid in the limit of weak disorder. They argued that the logarithmic nature of the series indicates that $\tilde{\alpha}$ should display critical behavior close to the mobility edge and conjectured the critical exponent by exponentiating the logarithmic series.

Kirkpatrick and Belitz calculated $1/k_F l^2$ corrections to $\tilde{\alpha}$ and showed that the logarithmic series did not exponentiate, thereby ruling out the previously guessed exponent. The $1/k_F l$ expansion is therefore not powerful enough to elucidate the critical behavior of $\tilde{\alpha}$. The representation of the ultrasonic attenuation in terms of a local nonlinear $\sigma$-model composite operator allows us to analyze in detail the critical behavior of $\tilde{\alpha}$ at the mobility edge. We show that the exponent conjectured by Kotliar and Ramakrishnan for the orthogonal case is incorrect by a factor of 2 and we also obtain the critical exponent for the ultrasonic attenuation in the unitary case.

We analyze the Anderson problem in the presence of time-reversal symmetry and in its absence. The latter case, which corresponds to a unitary nonlinear $\sigma$ model, is particularly interesting since it clearly indicates that the conductivity has both a local and a nonlocal part. The local part is represented by the expectation value of a local composite operator, while the nonlocal part is given by a correlation function (at zero momenta) of composite operators involving derivatives. This correlation function seems to vanish to three-loop order in the orthogonal case but gives a nonzero result at two-loop order in the unitary case.

The setup of this paper is the following. In Sec. II we review the mapping of the localization problem onto the nonlinear $\sigma$ model. We carry out the derivation in the presence of sources to generate the conductivity and the ultrasonic attenuation coefficient. In Sec. III we discuss the operators representing the conductivity after having shown that they can be derived directly using gauge in-

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variance symmetries. In Sec. IV we study the critical behavior of the ultrasonic attenuation coefficient in the orthogonal and the unitary case. We conclude with a discussion of possible applications of this formalism to other problems.

A sketch of the method, and the analysis of the critical behavior of $\bar{\alpha}$ in the presence of time-reversal symmetry, was presented in a Rapid Communication.\textsuperscript{11} The representation of the nonlocal part of $\sigma$ as a correlation function and the analysis of the critical behavior of $\bar{\alpha}$ in the absence of time reversal invariance is new. Independently, Kirkpatrick and Belitz\textsuperscript{12} arrived at the same conclusions for the critical behavior of $\bar{\alpha}$ in the orthogonal case. We believe that our derivation has some technical advantages over theirs and we comment on these points in Sec. II.

II. NONLINEAR $\sigma$ MODEL IN THE PRESENCE OF SOURCES

The mapping of the Anderson localization problem onto the nonlinear $\sigma$ model, first proposed by Wegner,\textsuperscript{5} has been reviewed by several authors.\textsuperscript{6,13–16} Here we review this mapping to establish the notation, placing special emphasis on the sources that are needed to generate the current-current and stress-stress correlation functions. This section follows mainly the work of Pruiskin and Schaefer.\textsuperscript{16} For simplicity we consider the orthogonal case, and indicate the changes that are needed to treat the unitary symmetry.

We consider an Anderson Hamiltonian

$$H = H_0 + V(r).$$  

(2.1)

$H_0$ is a nonrandom Hamiltonian (in the orthogonal case $H_0 = -\nabla^2/2$ is real) and $V(r)$ is a Gaussian random potential with variance $\langle V(r)V'(r) \rangle = g \delta(r-r')$.

The Anderson transition can be studied from quenched averages of products of one-particle Green’s functions

$$G_p(r,r') = \left\langle \frac{1}{E_p - H} \right\rangle ,$$  

(2.2)

$$E_p = E - \frac{s_p}{2} \omega, \quad \text{Re} \omega > 0 ,$$  

(2.3)

with $s_p$ a two-component vector $s_1 = i, s_2 = -i$.

At fixed disorder, Green’s functions are conveniently expressed as functional averages of classical real fields

$$G_p(r,r') = \langle \phi_p^\alpha(r') \phi_p^\alpha(r) \rangle$$  

(2.4)

with respect to the weight $\exp(-\int d^4r \mathcal{L})$,

$$\mathcal{L}[\phi] = \frac{1}{2} \sum_{\alpha,p} s_p \phi_p^\alpha(r)[E_p - H_0 - V(r)] \phi_p^\alpha(r) .$$  

(2.5)

$\alpha$ is a replica index $\alpha = 1, \ldots, n$ which is introduced to carry out the quenched average over the disorder directly. The averaged products of Green’s functions are then given by expectation values of the fields $\phi$ with the weight $\exp(-\int d^4r \mathcal{L})$:

$$\langle \mathcal{L}[\phi] = \frac{1}{2} \sum_{\alpha,p} s_p \phi_p^\alpha(E_p - H_0) \phi_p^\alpha$$

$$- \frac{g}{8} \left\langle \left[ \sum_{\alpha,p} s_p \phi_p^\alpha(r) \phi_p^\beta(r) \right]^2 \right\rangle .$$  

(2.6)

In this paper we will be interested in the conductivity which is given by the Kubo formula

$$\sigma = \frac{\pi e^2}{V} \sum_{n,m} \left\langle \left\langle n \mid p_x \mid m \right\rangle \right\rangle^2 \delta(\epsilon_n - E) \delta(\epsilon_m - E) \right\rangle_{av} .$$  

(2.7)

$\langle n \rangle$ denotes an exact eigenstate of Hamiltonian (2.1) with eigenvalue $\epsilon_n$. $V$ is the volume of the system and the electronic mass has been assumed to be unitary. The brackets $\langle \mid \rangle_{av}$ denote the quenched average.

The ultrasonic attenuation coefficient $\tilde{\alpha}(\omega)$,\textsuperscript{17} giving the inverse of the scale over which a transverse acoustic wave of frequency $\omega$ decays, can be expressed in terms of a stress-stress correlation function $\tilde{\alpha}(\omega) = (\omega^2 / 2 \rho_s C^2) \alpha(\omega)$:

$$\alpha = \frac{\pi}{V} \sum_{n,m} \left\langle \left\langle n \mid p_x \mid m \right\rangle \right\rangle^2 \delta(\epsilon_n - E) \delta(\epsilon_m - E) \right\rangle_{av} .$$  

(2.8)

$\rho_s$ is the mass density of the solid and $C$ is the speed of sound.

Equations (2.7) and (2.8) are easily expressed\textsuperscript{18} in terms of the one-particle Green’s functions and their nonlinear $\sigma$-model representation:

$$\sigma = \frac{e^2}{4\pi V} \int d^4r d^4r' \left\langle \nabla_x \left[ G_2(r,r') - G_1(r,r') \right] \nabla_x \left[ G_2(r',r) - G_1(r',r) \right] \right\rangle_{av}$$

$$= \frac{e^2}{16\pi V} \int d^4r d^4r' \sum_{\alpha,\beta,\alpha',\beta'} \left\langle \phi_{p_1}^\alpha(r) \nabla_x \phi_{p_2}^\beta(r) \phi_{p_1'}^\alpha(r') \nabla_x \phi_{p_2'}^\beta(r') \right\rangle_{av} .$$  

(2.9)

$$\epsilon_{\alpha \beta} = \delta_{\alpha 1} \delta_{\beta 2} - \delta_{\alpha 2} \delta_{\beta 1}$$ is an antisymmetric tensor which selects two different replica indices. Notice that while the summation over the energy indices $p_1, p_1'$ is free, the only nonzero contribution comes from pairwise equal indices. The same reasoning shows that
\[
\alpha = \frac{1}{4\pi V} \int d^4r \int d^4r' \nabla_x \nabla_y \bigl[ (G_2(\mathbf{r}, \mathbf{r'}) - G_1(\mathbf{r}, \mathbf{r'})) \nabla_x \nabla_y [G_1(\mathbf{r'}, \mathbf{r}) - G_2(\mathbf{r'}, \mathbf{r})] \bigr] \cdot \epsilon_{\alpha \beta \delta \mu} \\
= \frac{1}{16\pi V} \int d^4r \int d^4r' \sum_{\alpha, \beta, \alpha', \beta', p_1, p_2, \rho_1, \rho_2} \left( \phi_{\alpha \beta}^p(r) \nabla_x \nabla_y \phi_{\alpha' \beta'}^{p'}(r') \epsilon_{\alpha \rho_1 \beta \rho_2} \phi_{\alpha' \rho_1}^{p'}(r') \nabla_x \nabla_y \phi_{\beta' \rho_2}^{p}(r') \epsilon_{\beta \rho_1 \beta' \rho_2} \right),
\]

with \( \epsilon_{\alpha \beta \delta \mu} = \delta_{\alpha \delta} \delta_{\beta \mu} + \delta_{\alpha \mu} \delta_{\beta \delta} \) a symmetric tensor.

Equations (2.10) and (2.12) can be derived by adding sources
\[
\frac{1}{2} J_c \sum_{\alpha, \beta, p, p'} \phi_{\alpha \beta}^p(r) \nabla_x \phi_{\alpha \beta}^{p'}(r) \epsilon_{\alpha \beta}^{c},
\]
\[
\frac{1}{2} J_u \sum_{\alpha, \beta, p, p'} \phi_{\alpha \beta}^p(r) \nabla_x \phi_{\alpha \beta}^{p'}(r) \epsilon_{\alpha \beta}^{u}
\]

leading to the Lagrangian in Eq. (2.6). In terms of these sources,
\[
\mathcal{L} = \sum_{\alpha, \beta, p, p'} \left[ \frac{1}{2} \phi_{\alpha \beta}^p(r) \nabla_s \left[ (E_p - H_0) \delta_{pp} \delta_{\alpha \beta} - Q_{pp}^a(r) \right] + J_c u \sum_{\alpha, \beta, p, p'} \left[ \nabla_s \phi_{\alpha \beta}^p(r) \epsilon_{\alpha \beta}^{c} + \frac{1}{4g} Q_{pp}^a(r) \phi_{\alpha \beta}^{p'}(r) \right] \right].
\]

The subscripts \( c \) and \( u \) stand for conductivity and ultrasound, respectively, \( D_c = \partial / \partial x, \quad D_u = (\partial / \partial x)(\partial / \partial y) \),
and the matrices \( L \) are defined by
\[
L_{cpp'} = \frac{1}{\sqrt{s_p}} \epsilon_{\alpha \beta}^{c} \frac{1}{\sqrt{s_{p'}}},
\]
\[
L_{upp'} = \frac{1}{\sqrt{s_p}} \epsilon_{\alpha \beta}^{u} \frac{1}{\sqrt{s_{p'}}}.
\]

Integrating out the \( \phi \) fields in \( \exp(-\int d^4r \mathcal{L}[\phi, Q]) \)
with \( \mathcal{L} \) given by Eq. (2.17) one arrives at the effective Lagrangian for the matrix field \( Q \):
\[
\mathcal{L}[Q] = \frac{1}{4g} \exp \left[ \frac{1}{2} \text{tr} Q^2 + \frac{1}{2} \text{tr} \ln[(E_p - H_0) \delta_{pp} \delta_{\alpha \beta} - Q_{pp}^a(r)] \right]
+ J_c u L_{cpp'} D_{c, u} \right].
\]

The saddle point (SP) of this Lagrangian is given by
\[
\langle Q_{pp}^a(r) \rangle_{sp} = g \langle \phi_{\alpha \beta}^p(r) \phi_{\alpha \beta}^{p'}(r) \rangle_{sp} \sqrt{s_p} \sqrt{s_{p'}}

\quad \cdot \frac{s_p}{2\tau} \delta_{pp} \delta_{\alpha \beta}^a,
\]

with \( 1/\tau = 2\pi \rho g, \quad p = \rho(E) \) being the density of states. In the limit \( \omega \rightarrow 0 \) the Lagrangian (2.17) is invariant under the global transformations
\[
\phi \rightarrow T \phi, \quad Q \rightarrow \frac{1}{\sqrt{s}} T^{-1} Q T^{-1} \frac{1}{\sqrt{s}}
\]

with \( T \) obeying \( T^2 T = s, \quad s_{pp}^a = \delta_{pp} \delta_{\alpha \beta} s_p \), i.e., \( T \) is a pseudosymmetric transformation. Therefore, the saddle-point equation, in the limit of vanishing frequency, admits a manifold of solutions generated by transfor-
where $G^0$ is the averaged one-particle Green's function. These terms can be expressed in terms of the variable

$$\hat{Q} = \sqrt{s} \mathbf{T} \mathbf{B}^{-1} \frac{1}{\sqrt{s}},$$

(2.25)

with $T = T'$ a pseudo-orthogonal transformation. To order $(\nabla \hat{Q})^2$ we have

$$L_2 + L_4 = \frac{\pi \sigma_0}{8} \text{tr} \nabla \hat{Q} \cdot \nabla \hat{Q},$$

(2.26)

with $\sigma_0$ equal to the bare conductivity (apart from the factor $e^2$)

$$\sigma_0 = -\frac{1}{2\pi} \int d^4 r' \frac{\partial}{\partial x} G^0_{12}(r - r') \frac{\partial}{\partial x'} G^0_{12}(r' - r'),$$

(2.27)

while

$$L_1 = -\frac{\alpha_0}{4} \pi \rho \text{tr}(s \hat{Q}).$$

(2.28)

$L_5$ is given by

$$L_5 = \frac{\pi j_c^2}{4} \alpha_0 \text{tr} \hat{Q} \hat{L}_c \hat{Q} \hat{L}_c,$$

(2.29)

$$L_5 = -\frac{\pi j_c^2}{4} \alpha_0 \text{tr} \hat{Q} \hat{L}_u \hat{Q} \hat{L}_u$$

(2.30)

for the conductivity and ultrasonic attenuation, respectively. In obtaining Eqs. (2.29) and (2.30) we used the fact that $L^2$ and then $\text{tr} L^2$ are zero. $\sigma_0$ and $\alpha_0$ are the bare Boltzmann values for the conductivity, and the ultrasonic attenuation respectively. $L_6$, which is of the order $\nabla \hat{Q}$ and is proportional to

$$\int d^4 r' \nabla G^0_{12}(r - r') D G^0_{12}(r' - r'),$$

vanishes for the ultrasound since $\nabla$ and $D$ have different symmetries. It is, however, nonvanishing for the conductivity and in this case we obtain

$$L_3 + L_6 = -j_c \frac{\pi}{2} \alpha_0 \text{tr}(\nabla \hat{Q}) \hat{Q} \hat{L}_c.$$

(2.32)

Combining equations (2.26) and (2.28) we find, in the absence of sources, the effective Lagrangian for the soft fluctuations

$$\mathcal{L} = \frac{\alpha_0}{8} \text{tr}(\nabla \hat{Q})^2 - \frac{\alpha_0}{4} \pi \rho \text{tr}(s \hat{Q}),$$

(2.33)

with $\hat{Q}$ defined in Eq. (2.25) obeying the constraint $\hat{Q}^2 = -I$. We can parametrize $\hat{Q}$ as

$$\hat{Q} = \begin{pmatrix} i(1 + Q Q^t)^{1/2} & Q \\ Q^t & -i(1 + Q Q^t)^{1/2} \end{pmatrix},$$

(2.34)

which automatically solves the constraint $\hat{Q}^2 = -I$.

The sources for the conductivity induce two terms in the nonlinear $\sigma$-model Lagrangian [see Eqs. (2.29) and (2.32)]:

$$\delta \mathcal{L}_c = \frac{\pi j_c^2}{4} \alpha_0 \sum_{p_1, p_2, p_3, p_4} \frac{1}{\sqrt{s_{p_1} s_{p_2} s_{p_3} s_{p_4}}} \left( \hat{Q}^{11}_{p_1 p_2} \hat{Q}^{21}_{p_1 p_4} + \hat{Q}^{12}_{p_1 p_2} \hat{Q}^{12}_{p_3 p_4} - \hat{Q}^{11}_{p_1 p_2} \hat{Q}^{12}_{p_3 p_4} - \hat{Q}^{22}_{p_1 p_2} \hat{Q}^{11}_{p_3 p_4} \right)$$

$$- \frac{\pi \alpha_0}{2} j_c \sum_{p_1, p_2, p_3, \beta} \frac{1}{\sqrt{s_{p_1} s_{p_2} s_{p_3}}} \left[ \left( \nabla_x \hat{Q}^{1\beta}_{p_1 p_2} \hat{Q}^{\beta 1}_{p_3 p_4} - \left( \nabla_x \hat{Q}^{1\beta}_{p_1 p_2} \hat{Q}^{\beta 1}_{p_3 p_4} \right) \right) \right].$$

(2.35)

These two terms will represent the local and nonlocal part of the conductivity, respectively. Only one local term is instead induced by the ultrasonic attenuation sources

$$\delta \mathcal{L}_u = -\frac{\pi j_c^2}{4} \alpha_0 \sum_{p_1, p_2, p_3, p_4} \frac{1}{\sqrt{s_{p_1} s_{p_2} s_{p_3} s_{p_4}}} \left( \hat{Q}^{21}_{p_1 p_2} \hat{Q}^{11}_{p_3 p_4} + \hat{Q}^{12}_{p_1 p_2} \hat{Q}^{12}_{p_3 p_4} + \hat{Q}^{11}_{p_1 p_2} \hat{Q}^{12}_{p_3 p_4} + \hat{Q}^{22}_{p_1 p_2} \hat{Q}^{11}_{p_3 p_4} \right).$$

(2.36)

In the unitary case, when time reversal symmetry is broken, the one-particle Green's functions are expressed in terms of complex fields so that Eqs. (2.4) and (2.5) are written as

$$G^0_{12}(r, r') = s_{p} \left( \phi_{p}^0(r')^* \phi_{p}^0(r) \right),$$

(2.37)

$$\mathcal{L}^U[\phi] = s_{p} \phi_{p}^0(r')^* \left[ E_p - H_0 - V(r) \right] \phi_{p}^0(r).$$

(2.38)

The sources in Eqs. (2.13) and (2.14) must be modified into

$$\sum_{\alpha, \beta, p, p'} \left[ \phi_{p}^0(r) \right]^* \nabla_x \phi_{p'}^0(r) e_{\alpha\beta},$$

(2.39)

$$\sum_{\alpha, \beta, p, p'} \left[ \phi_{p}^0(r) \right]^* \nabla_x \phi_{p'}^0(r) e_{\alpha\beta}^*,$$

(2.40)

with $e_{\alpha\beta} = j_{\alpha} \delta_{\alpha 1} \delta_{\beta 2} - j_{\alpha} \delta_{\alpha 2} \delta_{\beta 1}$ and $e_{\alpha\beta}^* = j_{\alpha} \delta_{\alpha 1} \delta_{\beta 2} + j_{\alpha} \delta_{\alpha 2} \delta_{\beta 1}$. Equations (2.15) and (2.16) now read

$$\sigma^U \frac{1}{4\pi V} \frac{\partial^2 Z}{\partial j_{\alpha} \partial j_{\alpha}^*} \bigg|_{j_{\alpha} = 0},$$

(2.41)

$$\alpha^U \frac{1}{4\pi V} \frac{\partial^2 Z}{\partial j_{\alpha} \partial j_{\alpha}^*} \bigg|_{j_{\alpha} = 0}.$$

(2.42)

By repeating the arguments leading to Eqs. (2.33), (2.35), and (2.36) we arrive at the unitary nonlinear $\sigma$ model

$$\mathcal{L}^U = \frac{\pi \sigma_0}{4} \text{tr}(\nabla \hat{Q})^2 - \frac{\alpha_0}{4} \pi \rho \text{tr}(s \hat{Q}),$$

(2.43)

with $\hat{Q}$ given by

$$\hat{Q} = \sqrt{s} U s U^{-1} \frac{1}{\sqrt{s}},$$

(2.44)

$U = U^\dagger$ being a pseudounitary transformation. The sources induce the terms

$$U = U^\dagger$$
\[ \delta \mathcal{L}^U_e = -\frac{\pi}{2} j_e^* j_e e^{-\phi} \sum_{p_1, p_2, p_3, p_4} \frac{1}{\sqrt{s_{p_1} s_{p_2} s_{p_3} s_{p_4}}} \left[ \mathcal{Q}_{p_1 p_2}^{11} \mathcal{Q}_{p_1 p_3}^{22} + \mathcal{Q}_{p_1 p_2}^{22} \mathcal{Q}_{p_1 p_3}^{11} \right] \\
\] \\
\[ \mathcal{L}^U_u = -\frac{\pi}{2} j_u^* \alpha_0 \sum_{p_1, p_2, p_3} \frac{1}{\sqrt{s_{p_1} s_{p_2} s_{p_3}}} \left[ \mathcal{Q}_{p_1 p_2}^{11} \mathcal{Q}_{p_1 p_3}^{22} + \mathcal{Q}_{p_1 p_2}^{22} \mathcal{Q}_{p_1 p_3}^{11} \right] \]

for the ultrasound attenuation.

In the unitary case the only difference between \( \sigma \) and \( \alpha \) is then given by the presence of the derivative term in Eq. (2.45). As we shall see, this term induces a nonlocal contribution to \( \sigma \), which accounts for the different scaling behavior of \( \sigma \) with respect to \( \alpha \).

Finally we note that we have derived \( \delta \mathcal{L}_e \) and \( \delta \mathcal{L}_u \) without invoking an expansion of the matrix \( \mathcal{Q} \) in powers of \( \phi \) [Eq. (2.34)]. This has to be contrasted with the approach of Ref. 12 which generated the operators in Eq. (2.36) by an expansion to lowest order in \( \phi \) without being able to consider that higher powers of \( \mathcal{Q} \) are present. Wegner has shown that the relevance of composite operators of the nonlinear \( \sigma \) model increases with the number of power of \( \mathcal{Q} \). Therefore it is crucial to identify the composite operator containing the highest number of \( \mathcal{Q} \) fields to perform a scaling analysis.

### III. Conductivity

In this section we show that the operators representing the conductivity sources can be derived using arguments based on gauge invariance. We shall carry out the derivation for the orthogonal case, the unitary case being analogous. We note that the conductivity sources in Eq. (2.13) can be generated by applying the following pseudo-orthogonal transformation,

\[ \phi \rightarrow \tilde{T} \phi, \quad \tilde{T}(x) = \exp \left[ j_e \frac{1}{s} e^{\epsilon(x)} \right], \]

(3.1)

to the \( H_0 \) term in the Lagrangian Eq. (2.6): 

\[ -(\tilde{T} \phi \phi) H_0 \tilde{T} \phi = -\phi \phi H_0 \phi + \text{sources}. \]

The pseudo-orthogonality of \( \tilde{T} \) follows from \( (1/s) e^{\epsilon(x)} = -s [1/(1/s) e^{\epsilon(x)}] \).

Equation (2.13) is then a consequence of the identity

\[ \frac{\nabla^2}{2} j_e (1/s) e^{\epsilon(x)} = \left[ j_e (1/s) e^{\epsilon(x)} \right] \frac{1}{s} \nabla \phi + \frac{1}{s} \nabla^2, \]

(1/s) \( e^{\epsilon(x)} \) being zero.

\[ \mathcal{L}[\phi, \mathcal{Q}] \text{ in Eq. (2.17) has to be transformed into} \]

\[ \mathcal{L} = \frac{\pi \sigma_0}{8} \text{tr}(\nabla \mathcal{Q})^2 - \frac{\omega_0}{4} \mathcal{Q} \mathcal{Q} \text{tr}(\mathcal{Q}) \]

\[ = \frac{\pi \sigma_0}{8} \text{tr}(\nabla \mathcal{Q})^2 - \frac{\omega_0}{4} \mathcal{Q} \mathcal{Q} \text{tr}(\mathcal{Q}) + \frac{\pi \sigma_0}{8} \mathcal{Q} \mathcal{Q} \text{tr}(\mathcal{Q}) \]

+ \frac{\omega_0}{4} \mathcal{Q} \mathcal{Q} \text{tr}(\mathcal{Q}), \]n

(3.5)

where we have used \( \text{tr}(\nabla \mathcal{Q}) = \text{tr}(\mathcal{Q}) \) and

\[ \nabla_x \left[ \nabla_x \mathcal{Q} \frac{1}{s} \mathcal{Q} \mathcal{Q} \right] \]

\[ = \nabla_x \mathcal{Q} \frac{1}{s} \mathcal{Q} \mathcal{Q} \mathcal{Q} \frac{1}{s} \mathcal{Q} \]

\[ = \nabla_x \mathcal{Q} \frac{1}{s} \mathcal{Q} \mathcal{Q} \mathcal{Q} \frac{1}{s} \mathcal{Q} \]

By integrating out the \( \mathcal{Q} = \tilde{T} \phi \) fields and repeating the procedure leading to the nonlinear \( \sigma \)-model Lagrangian, Eq. (2.33), we obtain (with \( \mathcal{Q} = 2 \tau \mathcal{Q} \))
Equation (2.35) for \( \delta L_e \) is then recovered [see also Eqs. (2.29) and (2.32)]. We have therefore obtained that the sources term \( \delta L_e \) can be generated by applying the local gauge transformation in Eq. (3.1) to the gradient part of the nonlinear \( \sigma \)-model Lagrangian in the absence of sources, Eq. (2.33). We note that local gauge transformations when applied to the full Lagrangian in Eqs. (2.6) or (2.33) leave \( Z \) invariant. Physically this means that a gauge transformation introduces a vector field (coupled to the current) and a scalar field (coupled to the density) which exactly compensate each other. Ward identities are usually derived from this invariance. In the present case we have no action of the gauge transformation on the \( \omega \) term (the density term) so that we are left with an uncompensated vector field which acts as a real source.

We shall now discuss the critical behavior of the conductivity in the orthogonal and unitary case in the framework introduced in the previous section and confirmed by the above analysis.

In the orthogonal case, according to Eqs. (2.15) and (2.35) we have

\[
\sigma = \frac{\sigma_0}{V} \int d^4r \langle O_o(r) \rangle + \frac{\pi \sigma_0^2}{16V} \int d^4r d^4r' \langle O_1(r)O_1(r') \rangle,
\]

where

\[
O_o = -\frac{1}{4} \sum_{p,p'} (-1)^{p+p'} (\hat{Q}_{pp}^{12} \hat{Q}_{pp'}^{12} + \hat{Q}_{pp}^{21} \hat{Q}_{pp'}^{21}) - \hat{Q}_{pp}^{11} \hat{Q}_{pp'}^{22} - \hat{Q}_{pp}^{22} \hat{Q}_{pp'}^{11}),
\]

\[
O_1 = \sum_{p_1,p_2,p_3,\beta} \frac{1}{\sqrt{S_{p_1}S_{p_2}}} \left[ (\nabla_x \hat{Q}_{p_1p_2}^{1\beta} \hat{Q}_{p_2p_3}^{1\beta}) - (\nabla_x \hat{Q}_{p_1p_2}^{2\beta} \hat{Q}_{p_2p_3}^{2\beta}) \right].
\]

Equation (3.6) shows that the conductivity has both a local and a nonlocal part. The local part is represented by the expectation value of the composite operator \( O_o \). The nonlocal part is the correlation function of the operator \( O_1 \). Though \( O_1 \) includes one derivative, the presence of an additional space integration makes, in principle, the nonlocal part as relevant as the local one.

The local operator \( O_o \) in Eq. (3.7) coincides with the antisymmetric part \( O_A \) of the operator considered by Wegner, with the trace condition \( \sum_{ijkl} v_{ijkl} = 0 \), the index \( i \) combining a replica and energy index.

The scaling properties of \( O_A \) have been studied by Wegner in a 2 + \( \epsilon \) expansion. Up to three-loop order \( \langle O_A \rangle \sim \epsilon^{-3/4} \) with \( \lambda = 1.7,21 \). At four-loop order a correction \( \delta x_A = -\frac{1}{3} \epsilon^4 \) appears. \( \epsilon = (E - E_c) / \nu \) is the localization length, \( E - E_c \) being the distance from the mobility edge. Up to four-loop order \( \nu \) is given by \( \nu = 1/\epsilon + O(\epsilon) \). On the other hand, \( \sigma \) (being related via the Einstein relation to the diffusion coefficient, i.e., to the running coupling of the nonlinear \( \sigma \) model) exactly behaves as \( \epsilon^{-2} \). This means that the nonlocal part vanishes up to three-loop order, while it should contribute at four-loop order. We have explicitly checked that the nonlocal contribution to \( \sigma \) disappears up to two-loop order (see Appendix).

In the unitary case Eq. (3.6) is modified into [see Eq. (2.45)]

\[
\sigma = \frac{\sigma_0}{V} \int d^4r \langle O_o^U(r) \rangle - \frac{\pi \sigma_0^2}{4V} \int d^4r d^4r' \langle O_1^U(r)O_1^U(r') \rangle,
\]

with

\[
O_o^U = -\frac{1}{4} \sum_{p,p'} (-1)^{p+p'} (\hat{Q}_{pp}^{11} \hat{Q}_{pp'}^{22} + \hat{Q}_{pp}^{22} \hat{Q}_{pp'}^{11}),
\]

\[
O_1^U = \sum_{p_1,p_2,p_3,\beta} \frac{1}{\sqrt{S_{p_1}S_{p_2}}} \left[ (\nabla_x \hat{Q}_{p_1p_2}^{1\beta} \hat{Q}_{p_2p_3}^{1\beta}) - (\nabla_x \hat{Q}_{p_1p_2}^{2\beta} \hat{Q}_{p_2p_3}^{2\beta}) \right].
\]

In the present case, the nonlocal part gives a nonzero result already at two-loop order [Eq. (A19)]

\[
\frac{\delta \sigma}{\sigma_0}_{NL} = -\frac{t^2}{d} \left( \frac{2\epsilon^{\nu/2}}{\epsilon} \right), \quad d = 2 + \epsilon,
\]

with \( t = 1/4\pi^2 \sigma_0 \). From the local part, one instead obtains (see Appendix and next section)

\[
\frac{\sigma}{\sigma_0}_L = 1 + \frac{t^2}{2} \left( \frac{2\epsilon^{\nu/2}}{\epsilon} \right), \quad d = 2 + \epsilon.
\]

By summing Eqs. (3.12) and (3.13) one recovers the well-known two-loop result for the conductivity in the unitary case

\[
\frac{\sigma}{\sigma_0} = 1 + \frac{t^2}{\epsilon} \epsilon^{\nu},
\]

which leads to \( t^* = \sqrt{\epsilon / 2} \), \( \nu = 1/2\epsilon \).

IV. ULTRASONIC ATTENUATION

In this section we shall analyze the critical behavior of the ultrasonic attenuation coefficient in both the orthogonal and the unitary case.

In the orthogonal case, according to Eqs. (2.36) and (2.16), we can write

\[
\alpha = \alpha_0 \int d^4r \langle O_u(r) \rangle
\]

\[
O_u = -\frac{1}{4} \sum_{p,p'} (-1)^{p+p'} (\hat{Q}_{pp}^{12} \hat{Q}_{pp'}^{12} + \hat{Q}_{pp}^{21} \hat{Q}_{pp'}^{21}) + \hat{Q}_{pp}^{11} \hat{Q}_{pp'}^{22} + \hat{Q}_{pp}^{22} \hat{Q}_{pp'}^{11}.
\]

Unlike the conductivity \( \sigma \) can therefore be expressed as the expectation value of a local composite operator only.

To analyze the scaling behavior of \( \alpha \) we have to express the operator in Eq. (4.2) in terms of scaling operators. In the orthogonal case the scaling operators were calculated by Wegner. Besides the antisymmetric
operator $O_A$ defined in Eq. (3.7) one needs to introduce the traceless symmetric operator $O_s$:

$$O_s = -\frac{1}{4} \sum_{p,p'} ( -1)^{p+p'} [ 2 (\hat{Q}_{pp}^{12} \hat{Q}_{pp}^{12} + \hat{Q}_{pp}^{21} \hat{Q}_{pp}^{21} )$$
$$+ \hat{Q}_{pp}^{11} \hat{Q}_{pp}^{22} + \hat{Q}_{pp}^{22} \hat{Q}_{pp}^{11} ] .$$ (4.3)

$O_s$ is invariant under any permutation of the four indices, while $(\pi_{13} \pi_{14}) O_A = - O_A$ where $\pi_{13} (\pi_{14})$ interchanges the first and the third (fourth) index. The decomposition of $O_u$ in terms of $O_s$ and $O_A$ gives

$$O_u = \frac{3}{4} O_s + \frac{1}{4} O_A .$$ (4.4)

In the critical region $7, 21, 22$

$$\langle O_A \rangle = \xi^{-x_A} f_A (\omega \xi^d) ,$$

$$\langle O_s \rangle = \xi^{-x_s} f_s (\omega \xi^d) ,$$ (4.5)

with $x_A = -\frac{3}{2} \xi(3) \epsilon^d + O(\epsilon^5), \quad x_s = -\frac{5}{2} \xi(3) \epsilon^d + O(\epsilon^5)$. At criticality, the scaling function $f_A(x) \sim x^{-d/2}, x \rightarrow \infty$ and one obtains

$$\langle O_A \rangle \sim \omega^{-x_A / d} ,$$

$$\langle O_s \rangle \sim \omega^{-x_s / d} .$$ (4.6)

The ultrasound attenuation coefficient is determined by the symmetric operator which, having a negative $x_s$, is more relevant than the antisymmetric part.

The above value of $x_s$ implies

$$\alpha \sim \xi^{2 - (3/2) \xi(3) \epsilon^d}$$ (4.7)

close to criticality and $\alpha(\omega) \sim \omega^{\xi}$ with

$$\xi = -\frac{2 \epsilon}{2 + \epsilon} + \frac{\xi(3) \epsilon^d}{2 + \epsilon}$$ (4.8)

at the mobility edge. Since $\alpha$ is given by a sum of scaling operators, the exponent $\xi$ cannot be computed by simply exponentiating the perturbative logarithmic series. The perturbative series for $O_s$ and $O_A$ exponentiate separately at the fixed point but the series for $O_s + O_A$ of course does not. In fact expanding the exponentials in Eq. (4.6) we find

$$\frac{\alpha}{\alpha_0} = 1 - \frac{\xi}{2} \ln(\omega \tau) + \frac{1}{6} \xi^2 [\ln(\omega \tau)]^2 .$$ (4.9)

which coincides with the perturbative result of Kirkpatrick and Belitz.10

We use the same analysis to study the ultrasonic attenuation in the unitary case. From Eq. (2.46) we have

$$\alpha = \frac{1}{V} \int d^d r \langle O_u^{(d)}(r) \rangle ,$$

$$O_u^{(d)}(r) = -\frac{1}{4} \sum_{p,p'} ( -1)^{p+p'} [ 2 (\hat{Q}_{pp}^{12} \hat{Q}_{pp}^{12} + \hat{Q}_{pp}^{21} \hat{Q}_{pp}^{21} )$$
$$+ \hat{Q}_{pp}^{11} \hat{Q}_{pp}^{22} + \hat{Q}_{pp}^{22} \hat{Q}_{pp}^{11} ] .$$ (4.10)

One could expect that expression (4.2) reduces to expression (4.10) in the unitary nonlinear $\sigma$ model. In fact, $Q$ being now a complex matrix, the expectation values $\langle \hat{Q}_{pp}^{12} \hat{Q}_{pp}^{12} \rangle, \langle \hat{Q}_{pp}^{21} \hat{Q}_{pp}^{21} \rangle$ which correspond to particle-channel noncritical. The scaling operators in the unitary case were worked out by Pruisken,25

$$O_u^{(d)} = -\frac{1}{4} \sum_{p,p'} ( -1)^{p+p'} [ 2 (\hat{Q}_{pp}^{12} \hat{Q}_{pp}^{12} + \hat{Q}_{pp}^{21} \hat{Q}_{pp}^{21} )$$
$$+ \hat{Q}_{pp}^{11} \hat{Q}_{pp}^{22} + \hat{Q}_{pp}^{22} \hat{Q}_{pp}^{11} ] .$$ (4.11)

They are related to the symmetric and the antisymmetric representation of the permutation group.

Pruisken has shown that

$$\langle O_A \rangle \sim \xi^{-x_A} ,$$

$$\langle O_s \rangle \sim \xi^{-x_s} ,$$ (4.13)

with $x_A = -\sqrt{2} \epsilon$ and $x_s = \sqrt{2} \epsilon$. $\xi = (E - E_0)^{-\nu}$ and $\nu = 1/2$. Equations (4.13) and scaling arguments imply that at the mobility edge

$$\langle O_u^{(d)} \rangle \sim -\xi \sqrt{2} \epsilon / (2 + \epsilon) ,$$

$$\langle O_s^{(d)} \rangle \sim -\xi \sqrt{2} \epsilon / (2 + \epsilon) .$$ (4.14)

In terms of the scaling operators in Eq. (4.13), $\alpha$ is given by

$$\alpha = \frac{\alpha_0}{2} + \langle O_u^{(d)} + \langle O_A^{(d)} \rangle \rangle .$$ (4.15)

We do not expect the perturbative evaluation of $\alpha$ to exponentiate at the fixed point. In fact, Eqs. (4.15) and (4.14) imply

$$\alpha = \frac{\alpha_0}{2} ( e^{1/2} - 1 ) + e^{1/2}$$

$$\alpha = \alpha_0 ( 1 + \frac{1}{2} \epsilon )$$ (4.16)

which is the perturbative result of Kirkpatrick and Belitz at the unitary fixed point $(1/2 \pi E_T)^* = \sqrt{2} / 2$.

The scaling behavior of the ultrasonic attenuation is determined by the most relevant operator. We have therefore

$$\alpha \sim \xi \sqrt{2} \epsilon / (2 + \epsilon) .$$ (4.17)

V. CONCLUSIONS

In this paper we studied the consequences of representing transport coefficients in terms of the nonlinear $\sigma$ model. The ultrasonic attenuation coefficient is given by the expectation value of a local operator. This allows us to derive its critical behavior close to the Anderson transition in the orthogonal and unitary cases. Perturbative results9,10 are shown to be consistent with our analysis.

Unlike the ultrasonic attenuation, the conductivity has two different contributions, a local part which is represented as the expectation value of the local an-
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APPENDIX: TWO-LOOP CALCULATION OF THE CONDUCTIVITY IN THE UNITARY CASE

We explain the details of the evaluation of the correlation function of the operator $O^{12}$ [Eq. (3.11)] and the expectation value of $O_{12}^U$ [Eq. (3.10)]:

$$O^{12} = \sum_{p_{1},p_{2},p_{3},p_{4}} \frac{1}{\sqrt{p_{1}p_{2}p_{3}p_{4}}} (\nabla_{x_{1}} Q_{p_{1}p_{2}}) (\nabla_{x_{2}} Q_{p_{3}p_{4}}) ,$$
$$O_{12}^U = -\frac{1}{\bar{\tau}} \sum_{p,p'} (-1)^{p+p'} (\nabla_{x_{1}} Q_{p_{1}p}^{\mu} + \nabla_{x_{2}} Q_{p_{2}p'}^{\mu} + \nabla_{x_{1}} Q_{p_{1}p'}^{\mu} + \nabla_{x_{2}} Q_{p_{2}p}^{\mu}) ,$$

with respect to the nonlinear $\sigma$-model Lagrangian.

The expression for $\langle O^{12}O^{21}(r') \rangle$ can then be simplified to

$$\langle O^{12}(r)O^{21}(r') \rangle = \frac{1}{\bar{\tau}} \sum_{\beta,\gamma} \langle Q_{1\beta}(r) \nabla_{x} Q_{1\gamma}^{\mu}(r) \rangle \langle Q_{2\gamma}(r') \nabla_{x} Q_{2\beta}^{\mu}(r') \rangle ,$$

with

$$\frac{1}{\bar{\tau}} = \frac{\sigma_{0}}{2}, \quad \Omega = \omega \pi \rho \bar{\tau} .$$

The matrix $Q$ in the unitary nonlinear $\sigma$ model is parametrized by a complex matrix $Q = \bar{Q}^{-1}$:

$$Q = \begin{pmatrix} i(1 + \bar{Q}Q) & Q \cr Q^{\dagger} & -i(1 + \bar{Q}Q) \end{pmatrix} ,$$

$$\bar{Q} = \begin{pmatrix} i(1 - Qar{Q})^{1/2} \cr Q^{\dagger} \end{pmatrix} .$$

The lowest-order vertex (Fig. 1) is obtained by substituting Eq. (A5) into Eq. (A3) and is given by

$$\mathcal{L}_{\mathcal{Y}} = \frac{1}{4\bar{\tau}} \sum_{a,b,y} \nabla(Q^{ab}) \cdot \nabla(Q^{\gamma y} Q_{ab}^{\star}) ,$$

while the free part,

$$\mathcal{L}_{\mathcal{Y}} = \frac{1}{\bar{\tau}} \sum_{a,b} \{ (\nabla Q^{ab}) \cdot (\nabla Q^{ab}) + \Omega Q^{ab} Q_{ab}^{\star} \} ,$$

generates the propagator $\langle Q_{ab}^{\star}(k)Q_{ab}(k) \rangle = \bar{\tau}/(k^4 + \Omega)$ which we represent by a double line (Fig. 2).

The nonlinear correction to the conductivity in Eq. (3.9) is given by

$$\delta \sigma_{NL} = -\frac{\pi}{4} \sigma_{0}^{2} \bar{\tau} \int d^4r \langle O^{12}(r)O^{21}(0) \rangle .$$

Inserting Eq. (A5) into Eq. (A1) we find the form of the operator $O^{12}$ in terms of $Q$. It is composed of two parts: $O^{12}_{++}$ [Eq. (A1) with $p_{1} = p_{2}$] and $O^{12}_{+-}$ [Eq. (A1) with $p_{1} = -p_{2}$], which gives rise to vertices with an even and an odd number of legs, respectively. In the limit $n \to 0$, up to two-loop order, nonzero contributions to $\delta \sigma_{NL}$ come only from $O^{12}_{+-}$ with

$$O^{12}_{+-} = \frac{1}{2} \sum_{\gamma,\beta} \{ (\nabla_{x} Q_{1\beta}^{\gamma}) Q_{2\beta}^{\gamma} Q_{1\gamma}^{2\mu} + Q_{1\beta}^{\gamma} Q_{2\beta}^{\gamma} (\nabla_{x} Q_{1\gamma}^{2\mu}) \}
- (\nabla_{x} Q_{1\beta}) Q_{2\gamma}^{\gamma} Q_{1\gamma}^{2\mu} - Q_{1\beta}^{\gamma} Q_{2\beta}^{\gamma} (\nabla_{x} Q_{1\gamma}^{2\mu}) .$$

The expression for $\langle O^{12}O^{21} \rangle$ can then be simplified to
Equation (A10) generates the two diagrams in Fig. 3. The first one (a) is proportional to $n^2$ while the second one (b) gives
\[ J = \frac{1}{d} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{q^2}{(q^2 + \Omega)(k^2 + \Omega)(q + k)^2 + \Omega)} . \] (A12)

In dimensional regularization one can shift momenta and one finds that the singular part (containing poles in $\epsilon$) of $J$ can be expressed as
\[ J = \frac{1}{d} I^2 , \] (A13)
with
\[ I = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \Omega} \sim -\frac{1}{2\pi\epsilon} \Omega^{\epsilon/2} \] (A14)
in the limit $\epsilon \to 0$. This gives
\[ \left[ \frac{\delta \sigma}{\sigma_0} \right]_{\text{NL}} = -\frac{1}{4} \frac{\hat{t}^2 I^2}{d} . \] (A15)

The local contribution to the conductivity is evaluated along the same lines from Eq. (3.9):
\[ \left[ \frac{\delta \sigma}{\sigma_0} \right]_{L} = \langle O^U \rangle . \] (A16)

To two-loop order one obtains
\[ \langle O^U \rangle = \frac{1}{8} \sum_{\gamma, \beta} \langle Q^{\beta \gamma} Q^{\gamma} Q^{\gamma} \rangle . \] (A17)

The nonzero contraction is represented by the diagram in Fig. 4, which gives
\[ \langle O^U \rangle = \frac{1}{8} \hat{t}^2 I^2 . \] (A18)

By using Eq. (A14) and the definition $t = \hat{t}/8\pi = 1/4\pi^2\sigma_0$ one eventually finds
\[ \left[ \frac{\delta \sigma}{\sigma_0} \right]_{\text{HL}} = -t^2 \frac{2}{\epsilon^2} \Omega^\epsilon \left[ \frac{2}{d} \right] . \] (A19)
\[ \left[ \frac{\delta \sigma}{\sigma_0} \right]_{L} = t^2 \frac{2}{\epsilon^2} \Omega^\epsilon . \] (A20)

Together they combine to give the expression
\[ \sigma = \sigma_0 \left[ 1 + \frac{t^2}{\epsilon} \Omega^\epsilon \right] , \] (A21)

which is the regularized expression of the conductivity in the unitary case. The $\beta$ function for the unitary case\textsuperscript{23,\textsuperscript{24}}
\[ \beta(t) = -\epsilon t + 2t^3 \] (A22)
is then recovered.

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\textsuperscript{6}Permanent address: Dipartimento di Fisica, Università “La Sapienza,” I-00185 Roma, Italy.


\textsuperscript{4}E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V.
A derivation of the conductivity in terms of nonlinear $\sigma$-model composite operators was obtained by B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, Pis'ma Zh. Eksp. Teor. Fiz. 43, 342 (1986) [JETP Lett. 43, 441 (1986)] using the framework of Ref. 6. The nonlocal contribution was however not included in their work. A similar derivation was first given by A. M. Pruisken, in The Context of the Quantum Hall Effect, Vol. 61 of Springer Series in Solid-State Science, edited by B. Kramer, G. Bergmann, and Y. Bruynseraede, (Springer, Berlin, 1985). See also the Quantum Hall Effect (Springer-Verlag, Berlin, 1986).


Equations (2.7) and (2.8) are written in the limit $\omega \to 0$. Equations (2.9) and (2.11) allow the evaluation of $\sigma$ and $\alpha$ at finite (small) frequency by letting $\omega \to -i\omega + 0^+$ in Eq. (2.3). The factor $1/4g$, instead of $1/2g$, in the last term of Eq. (2.17) is introduced by applying the Hubbard-Stratonovich transformation in the momenta space, with the constraint that small momenta must appear in the $\mathcal{O}$ matrices:

$$\int d^d r \, \phi_\sigma^\dagger(\mathbf{r}) \phi_\sigma^\dagger(\mathbf{r}) \phi_\sigma(\mathbf{r}) \phi_\sigma^\dagger(\mathbf{r})$$

$$\approx \frac{1}{V} \sum_{q, k, k'} \left[ \phi_\sigma^\dagger(k + q) \phi_\sigma(k - q) \phi_\sigma(-k') \phi_\sigma(k) + \phi_\sigma^\dagger(k + q) \phi_\sigma(-k') \phi_\sigma(k) \phi_\sigma(-k') \right] ,$$

with $q$ small.

We use the Ward identity $\int d^d r' \nabla_x G_\sigma^\dagger(r' - r') \nabla_x G_\sigma^\dagger(r' - r) = G_\sigma^\dagger(0)$; cf. Ref. 16.


In deriving this conclusion we assume that both results given by Refs. 22 and 23 are correct.